Unit 3

Second-order differential equations

Introduction

In this unit we move from first-order differential equations to second-order differential equations, that is, differential equations involving a second (but no higher) derivative. Examples of such equations are

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^x \quad \text{and} \quad 3\frac{d^2y}{dx^2} + y = x\sin x.$$

The *order* of a differential equation is defined in Unit 2. A second-order differential equation may or may not include a first derivative.

Second-order differential equations play a central role in the physical sciences. They are found, for example, in laws describing mechanical systems, wave motion, electric currents and quantum phenomena.

To take a simple case, consider a particle of mass m that moves in one dimension along the x-axis. At any given time t, the particle's position is $x(t)$, and its velocity and acceleration are given by the derivatives dx/dt and d^2x/dt^2. There are no general laws for the position or velocity of the particle, but there is a very important law for its acceleration: *Newton's second law* tells us that

Don't worry if you have not met Newton's second law before: the essential aims of this unit do not rely on it.

$$\text{mass} \times \text{acceleration} = \text{force},$$

which implies that

$$m\frac{d^2x}{dt^2} = F, \tag{1}$$

where F is the force acting on the particle. The force need not be constant, and may vary with the position x or the velocity dx/dt of the particle. So, depending on the precise details, we get a second-order differential equation for x as a function of t, and the solution of this equation tells us how the particle can move.

The system known as a *simple harmonic oscillator* provides a good example. Here, a particle of mass m is suspended at the lower end of a spring that is attached to a fixed support (Figure 1). The particle moves up and down along a vertical x-axis, subject to a force F provided by the spring and gravity. If the system is left to settle, the particle comes to rest at a point of equilibrium, which we label $x = 0$. Because the particle does not spontaneously move away from this position, we can infer that $F = 0$ when $x = 0$.

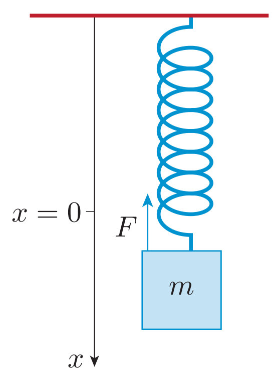

Figure 1 A particle of mass m moves along the x-axis subject to a force F provided by a spring and gravity

When the particle is displaced from $x = 0$, the force F tends to draw the particle back towards $x = 0$. We consider the case where the force is proportional to the displacement from equilibrium, and take

$$F = -kx, \tag{2}$$

where k is a positive constant. The negative sign in this equation ensures that the force always acts in a direction that tends to restore the particle to its equilibrium position.

Putting equations (1) and (2) together, we get the differential equation

$$m\frac{d^2x}{dt^2} = -kx.$$

Recalling that $k > 0$ and $m > 0$, we can also express this as

$$\frac{d^2x}{dt^2} = -\omega^2 x, \tag{3}$$

where $\omega = \sqrt{k/m}$ is a positive constant. Equation (3) is called the *equation of motion* of a simple harmonic oscillator. It is a second-order differential equation whose solution tells us how the particle can move.

This unit develops systematic techniques to solve equations like this. For the moment, we will simply guess the solution and check that it works. You know that

$$\frac{d}{dt}(\sin t) = \cos t \quad \text{and} \quad \frac{d}{dt}(\cos t) = -\sin t,$$

so

$$\frac{d^2}{dt^2}(\sin t) = -\sin t \quad \text{and} \quad \frac{d^2}{dt^2}(\cos t) = -\cos t.$$

In other words, taking the second derivative of a sine or cosine function gives the same function back again, but with a minus sign. This is very close to the behaviour needed to solve equation (3). We therefore try a function of the form

$$x(t) = C\sin(\omega t) + D\cos(\omega t), \tag{4}$$

where C and D are any constants, and ω is the constant in equation (3). Differentiating this function once, and then again, we get

$$\begin{aligned}\frac{dx}{dt} &= C\omega\cos(\omega t) - D\omega\sin(\omega t),\\ \frac{d^2x}{dt^2} &= -C\omega^2\sin(\omega t) - D\omega^2\cos(\omega t) = -\omega^2 x.\end{aligned}$$

So the function in equation (4) does indeed satisfy equation (3). In fact, it is the general solution of this differential equation.

Notice that our solution involves two constants, C and D, whose values are not specified. These constants have arbitrary values, and they are called *arbitrary constants*. It is typical for the general solutions of a second-order differential equation to have *two* arbitrary constants. The values of these constants depend on how the system is released, and you will see how they are determined later in this unit. To take a definite case, suppose that $C = 2$, $D = 3$ and $\omega = 4$, when measured in suitable units. Then we have the particular solution

$$x(t) = 2\sin(4t) + 3\cos(4t), \tag{5}$$

and this is plotted in Figure 2. The wiggles in this graph correspond to the oscillations performed by a system like the particle on the end of the spring in Figure 1.

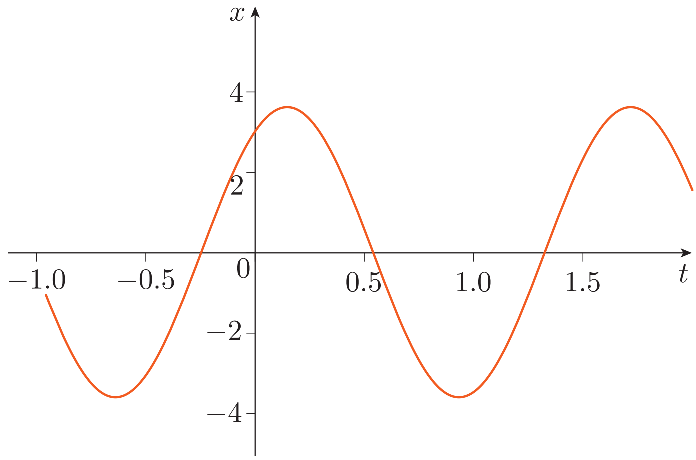

Figure 2 A graph of equation (5), which is a particular solution of equation (3)

The simple harmonic motion described by equation (4) continues forever, but we know from everyday experience that oscillations generally die away after a while. We can get a more realistic description by including an additional force in our model – one that will dampen the oscillations down. We take the additional force to be $-\gamma\, dx/dt$, where γ is a positive constant, leading to the differential equation

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}. \tag{6}$$

This is the equation of motion of a *damped harmonic oscillator*. Later in this unit you will see that (in appropriate circumstances) it has solutions that oscillate but diminish and eventually die away.

We can also add in another force, $f(t)$, which is applied to the particle by some external agency. We assume that this force is known directly as a function of time (and does not depend on the position or velocity of the particle.) Under these circumstances, Newton's second law leads to the differential equation

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + f(t), \tag{7}$$

and this is the equation of motion of a *forced damped harmonic oscillator*. If the external force is oscillatory, the response of the system may depend very sensitively on the frequency of the external force. This phenomenon is called *resonance*; it will be explored at the end of this unit.

Harmonic oscillators are everywhere

Harmonic oscillators play a central role in physics and its applications. If a system performs small oscillations about an equilibrium point, then it is generally a good approximation to model it as a harmonic oscillator, including the additional terms in equations (6) and (7) when necessary.

It should come as no surprise that the to-and-fro motion of a pendulum clock can be modelled by a harmonic oscillator. On a smaller scale, vibrating molecules and vibrating crystals are also modelled as harmonic oscillators.

Equations similar to (7) are also used to describe the oscillations of currents in electrical circuits that allow radios to be tuned to selected stations. Moreover, each frequency in an electromagnetic field can be regarded as a harmonic oscillator, and this is a key insight used in advanced physics when electromagnetic fields are treated quantum mechanically.

A more everyday example is provided by the suspension system of a mountain bike (or any other vehicle). The rider is protected from the vibrations caused by a rough track by a rugged suspension system, such as that shown in Figure 3, and this can also be modelled by equation (7).

Figure 3 The suspension system of a mountain bike is designed to absorb shocks. The rear part of the frame can rotate about a pivot point, compressing or extending the spring.

Study guide

This unit requires no previous knowledge beyond that needed for Unit 2, apart from some familiarity with complex numbers. The relevant material on complex numbers was reviewed in Unit 1.

The differential equations discussed in this unit all belong to one broad class: they are all *linear constant-coefficient second-order differential equations*. These equations play such an important role in physics and areas of applied mathematics that they easily deserve a unit to themselves.

Section 1 introduces some basic principles and terminology. Sections 2 and 3 give methods for finding the general solutions for our class of second-order differential equations: Section 2 covers so-called *homogeneous equations*, while Section 3 covers *inhomogeneous equations*. Section 4 then explains how extra information can be used to help us to select particular solutions that are appropriate in given situations.

For the most part, the unit presents the topic of second-order differential equations in purely mathematical terms, and its learning aims can be fully met without reference to physical laws. However, it is illuminating to see physical interpretations of the mathematical equations, so Subsection 2.4 and Section 5 revisit the oscillators considered in the Introduction.

Although the differential equations discussed in this unit and the last may seem to belong to a limited range of classes, in practice they encompass most of what a mathematical scientist needs to know about differential equations. The results obtained in this unit will be used again in Units 6 and 12.

1 Some preliminary remarks

As for first-order differential equations, second-order differential equations can be written using a variety of notations for functions and derivatives.

If t is the independent variable and y is the dependent variable, we can regard y as a function of t and write $y = f(t)$. More usually, however, we write $y = y(t)$, using the same symbol y for both the variable and the function. (The merits of this notation were discussed in Unit 1.) The first derivative of y with respect to t may be written as dy/dt, $\dot{y}$ or y', and the second derivative as d^2y/dt^2, $\ddot{y}$ or y''.

Of course, the independent variable is not always t, and the dependent variable is not always y!

This section makes some general comments that will be important for understanding the methods introduced in Sections 2 and 3.

1.1 Requirement for two arbitrary constants

In Unit 1 you saw that when we solve a first-order differential equation, we get a general solution containing one *arbitrary constant*. The value of the arbitrary constant is undetermined in general, but it can be found by using additional information provided by an initial condition. Here we consider the corresponding result for second-order equations. You saw in the Introduction that the differential equation for a simple harmonic oscillator (equation (3)) has a general solution (equation (4)) that contains two arbitrary constants. This turns out to be the general rule.

You can assume that the general solution of any second-order differential equation contains two arbitrary constants.

To further illustrate this point, consider the differential equation

$$\frac{d^2y}{dt^2} = a, \tag{8}$$

where t is the independent variable, and a is a given constant. This equation describes the motion of a particle with constant acceleration a. To take a definite case, we will measure distance in metres and time in seconds, and suppose that $a = 3$ in these units. Our differential equation then becomes

$$\frac{d^2y}{dt^2} = 3. \tag{9}$$

An equation like this can be solved by integrating both sides twice. The first integration gives

$$\frac{dy}{dt} = \int 3\,dt = 3t + C,$$

where C is an arbitrary constant. A second integration then gives

$$y(t) = \int (3t + C)\,dt = \tfrac{3}{2}t^2 + Ct + D, \tag{10}$$

where D is another arbitrary constant. Equation (10) is the *general solution* of the differential equation. It gives a formula that describes the collection of all possible solutions of the equation. As promised, this formula contains two arbitrary constants, C and D.

If we take definite values for C and D, we get a *particular solution* of the differential equation. For example, if we take $C = 2$ and $D = 3$, then the particular solution is

$$y(t) = \tfrac{3}{2}t^2 + 2t + 3.$$

In the context of straight-line motion, this solution tells us where the particle is located at each instant t. Moreover, differentiating both sides gives

$$\frac{dy}{dt} = 3t + 2,$$

so it also gives the velocity dy/dt of the particle at each instant t.

How do we determine the arbitrary constants? You have seen that a second-order differential equation has *two* arbitrary constants, so we need *two* pieces of information to determine them. A whole section of this unit (Section 4) is devoted to ways of finding the arbitrary constants, but we will make a brief comment now.

One way of finding the arbitrary constants is to use initial conditions. However, it is not enough to specify the value of the function $y(t)$ at a fixed time $t = t_0$. We also need extra information, and this can be provided by giving the value of the derivative dy/dt at the *same* fixed time, $t = t_0$.

Example 1

The following description of motion is based on units of metres and seconds. At $t = 0$, a car has initial position $y = 100$ and initial velocity $u = 4$. Between $t = 0$ and $t = 10$, the car travels with constant acceleration $a = 3$ along a straight road.

Find the particular solution of equation (9) in this case, and use it to predict the car's position and velocity at $t = 10$.

Solution

Following the above calculation, the general solution of equation (9) is

$$y(t) = \tfrac{3}{2}t^2 + Ct + D, \qquad \text{(Eq. 10)}$$

where C and D are arbitrary constants. The initial conditions are $y(0) = 100$ and $y'(0) = 4$. The first condition gives

$$100 = \tfrac{3}{2} \times 0^2 + C \times 0 + D = D.$$

The condition $y'(0) = 4$ could also be written as $\dot{y}(0) = 4$ or as $\left.\dfrac{dy}{dt}\right|_{t=0} = 4$.

Differentiating equation (10) gives $y'(t) = 3t + C$, so the second condition gives

$$4 = 3 \times 0 + C = C.$$

We conclude that the arbitrary constants have values $C = 4$ and $D = 100$. The required particular solution is therefore

$$y(t) = \tfrac{3}{2}t^2 + 4t + 100.$$

Differentiating this function gives the car's velocity as a function of time:

$$y'(t) = 3t + 4.$$

We obtain $y(10) = 290$ and $y'(10) = 34$. So at time 10 seconds, the car's position is 290 metres and its velocity is 34 metres per second.

The solution of second-order differential equations is rarely as easy as the solution of equation (9). In fact, the approach of repeated direct integration works only for equations of the form

$$\frac{d^2y}{dx^2} = f(x),$$

and requires that both integrations can be carried out.

Initial conditions and Newton's second law

You have seen that Newton's second law leads to second-order differential equations, and that the general solution of a second-order differential equation contains *two* arbitrary constants.

The need for two arbitrary constants connects to everyday experience. If you throw a stone upwards, and want to predict its future motion, it is not enough to know where the stone was released. You need to know both its initial position *and* its initial velocity. These two bits of information are sufficient to determine the two arbitrary constants in the general solution, and hence select a unique particular solution.

A single differential equation has a multitude of particular solutions, distinguished by different arbitrary constants. In the context of mechanics, this is a great unifying feature. We do not need different rules to explain the various motions of falling apples, cricket balls, asteroids or orbiting satellites. All these motions can be modelled by the same differential equation (obtained from Newton's second law and the law of gravitation). If the motions are different, it is because the initial conditions are different. In a nutshell: a differential equation provides unity, and initial conditions provide variety.

1.2 Linearity and superposition

This unit considers second-order differential equations that are *linear* and have *constant coefficients*. You met linear constant-coefficient equations in Unit 2 in the context of first-order differential equations. But what do the terms 'linear' and 'constant-coefficient' mean in the context of second-order equations? The answer lies in the following definitions.

Definitions

Compare the definitions for first-order equations in Unit 2. The important feature is the *linear* combination of y and its derivatives on the left-hand side.

- A second-order differential equation for $y = y(x)$ is **linear** if it can be expressed in the form
$$a(x)\,\frac{d^2y}{dx^2} + b(x)\,\frac{dy}{dx} + c(x)\,y = f(x),$$
where $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are given continuous functions, and $a(x)$ is not equal to the zero function.
- A linear second-order differential equation is said to be **constant-coefficient** if the functions $a(x)$, $b(x)$ and $c(x)$ are all constants, so that the equation is of the form
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{11}$$
where $a \neq 0$.

If $a = 0$, then the equation is first-order.

- A linear constant-coefficient second-order differential equation is said to be **homogeneous** if $f(x) = 0$ for all x, and **inhomogeneous** otherwise.

The term **non-homogeneous** is sometimes used instead of inhomogeneous.

Linear constant-coefficient second-order differential equations can be written in other ways. For example, we can divide equation (11) through by a to obtain an equation of the form

$$\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y = \phi(x),$$

where β and γ are constants, and this more closely resembles the definition of linear first-order differential equations from Unit 2.

Exercise 1

Consider the following second-order differential equations.

(i) $\dfrac{d^2y}{dx^2} = x^2$ (ii) $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = x^2$ (iii) $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$

(iv) $xy'' + x^2y = 0$ (v) $2y\dfrac{d^2y}{dx^2} + xy = 3\dfrac{dy}{dx}$ (vi) $2y\dfrac{d^2y}{dx^2} + 4y = 3\dfrac{dy}{dx}$

(vii) $2\dfrac{d^2t}{d\theta^2} + 3\dfrac{dt}{d\theta} + 4t = \sin\theta$ (viii) $\ddot{x} = -4t$ (ix) $\ddot{x} = -4x$

(a) For each equation, identify the dependent and independent variables.

(b) Which of the equations are linear?

(c) Which of the equations are linear and constant-coefficient?

(d) Which of the linear constant-coefficient equations are homogeneous?

The principle of superposition

We now introduce a key principle that will turn out to be extremely useful throughout this unit. The principle is a fundamental property of all linear differential equations, but we discuss it here in the context of linear constant-coefficient second-order equations.

Suppose that we have a solution $y_1(x)$ of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_1(x),$$

In principle, the coefficients a, b and c could be functions of x, but our interest in this unit is confined to the case where they are constants.

and a solution $y_2(x)$ of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_2(x).$$

Then the *principle of superposition* states that any linear combination $k_1\, y_1(x) + k_2\, y_2(x)$, where k_1 and k_2 are constants, is a solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k_1\, f_1(x) + k_2\, f_2(x). \tag{12}$$

It is not difficult to see why this is true. If we substitute $k_1y_1 + k_2y_2$ into the left-hand side of equation (12), we get

$$\begin{aligned}
&a\frac{d^2}{dx^2}(k_1y_1 + k_2y_2) + b\frac{d}{dx}(k_1y_1 + k_2y_2) + c(k_1y_1 + k_2y_2)\\
&\quad = a\left(k_1\frac{d^2y_1}{dx^2} + k_2\frac{d^2y_2}{dx^2}\right) + b\left(k_1\frac{dy_1}{dx} + k_2\frac{dy_2}{dx}\right) + c(k_1y_1 + k_2y_2)\\
&\quad = k_1\left(a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right) + k_2\left(a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right)\\
&\quad = k_1\, f_1(x) + k_2\, f_2(x),
\end{aligned}$$

as required.

This important result is summarised as follows.

The principle of superposition

If $y_1(x)$ is a solution of the linear second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_1(x),$$

and $y_2(x)$ is a solution of the linear second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_2(x)$$

(with the same left-hand side), then the function

$$y(x) = k_1\, y_1(x) + k_2\, y_2(x),$$

where k_1 and k_2 are any constants, is a solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k_1\, f_1(x) + k_2\, f_2(x).$$

The power of this principle is obvious: it enables us to find new solutions by adding together existing ones.

An important special case arises when $f_1(x) = f_2(x) = 0$. In this case we see that if $y_1(x)$ and $y_2(x)$ are both solutions of the homogeneous equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{13}$$

then any linear combination

$$y(x) = k_1\, y_1(x) + k_2\, y_2(x),$$

where k_1 and k_2 are constants, is also a solution of the *same* homogeneous equation (13).

A note on terminology

Like all human languages, the language of science and mathematics tends to evolve and fragment into dialects.

In physics, the *principle of superposition* is sometimes given a meaning that is slightly more restricted than that used here. It is said to apply when 'any linear combination of solutions of a given equation is also a solution of the *same* equation'. This restricted form of the principle applies to any homogeneous equation like (13), but not to inhomogeneous ones. You may meet this alternative usage in other Open University texts, but the more general form of the principle, as given in the box above, is what is needed in this unit.

2 Homogeneous differential equations

This section develops a method for finding the *general solutions* of *homogeneous* linear constant-coefficient second-order differential equations. Section 3 will consider the general solutions of inhomogeneous equations, and Section 4 will discuss how particular solutions are selected in given situations.

Before the method for homogeneous equations is described in detail, it is helpful to look at two simple cases.

2.1 Two simple cases

In this section, we return to the equation of motion of a simple harmonic oscillator. Before tackling this, however, we will solve the closely-related equation

$$\frac{d^2y}{dt^2} - \omega^2 y = 0, \tag{14}$$

where ω is a given positive constant. This differs from the equation for a simple harmonic oscillator only in the sign of the coefficient of y, which is negative in this case.

Our method of solution is very simple. We notice that a function $y = e^{\lambda t}$, where λ is any constant, can be differentiated, and then differentiated again, to give

$$\frac{dy}{dt} = \lambda e^{\lambda t} \quad \text{and} \quad \frac{d^2y}{dt^2} = \lambda^2 e^{\lambda t}.$$

If we substitute this function into the differential equation (14), we get

$$\lambda^2 e^{\lambda t} - \omega^2 e^{\lambda t} = 0.$$

The exponential factors can be cancelled (because they are never equal to zero) and we are left with a simple algebraic equation for λ:

$$\lambda^2 - \omega^2 = 0.$$

This equation has two solutions: $\lambda = \omega$ and $\lambda = -\omega$. So we have found two distinct solutions of equation (14), namely

$$y = e^{\omega t} \quad \text{and} \quad y = e^{-\omega t}.$$

The argument then goes as follows:

- The differential equation is homogeneous, so the principle of superposition guarantees that any linear combination of the solutions $y = e^{\omega t}$ and $y = e^{-\omega t}$ is also a solution of the differential equation. Hence the function

  $$y(t) = Ce^{\omega t} + De^{-\omega t}, \tag{15}$$

  where C and D are arbitrary constants, satisfies equation (14).
- Then remember that the general solution of a second-order differential equation contains two arbitrary constants. The function $y(t) = Ce^{\omega t} + De^{-\omega t}$ satisfies the differential equation and contains two arbitrary constants. This means that it is the *general solution* of equation (14). The arbitrary constants could take any values, but if we want our solution to be real-valued, they must be real.

Now we consider the equation of motion of a simple harmonic oscillator, which can be written in the form (from equation (3))

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \tag{16}$$

where ω is a given positive constant.

The tactics for solving this equation are just the same. We try a solution of the form $e^{\lambda t}$, where λ is an undetermined constant. Substituting this into the differential equation, we obtain

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0.$$

Then, cancelling the (non-zero) exponential factors, we get the algebraic equation

$$\lambda^2 + \omega^2 = 0.$$

The solutions of this equation are the complex numbers $\lambda = i\omega$ and $\lambda = -i\omega$, and corresponding to these we have two distinct solutions of equation (16):

$$y = e^{i\omega t} \quad \text{and} \quad y = e^{-i\omega t}.$$

Using the principle of superposition, and the fact that the general solution of a second-order differential equation contains two arbitrary constants, we can follow an argument like that given above to conclude that the general solution of equation (16) is

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}, \tag{17}$$

where A and B are arbitrary constants (which may be complex numbers in this case).

There is nothing wrong with this solution, but we can put it in a more familiar form by using Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Substituting $x = \omega t$ gives

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t),$$

while substituting $x = -\omega t$ gives

$$e^{-i\omega t} = \cos(-\omega t) + i \sin(-\omega t) = \cos(\omega t) - i \sin(\omega t).$$

We therefore see that the general solution is

$$\begin{aligned} y(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= A[\cos(\omega t) + i \sin(\omega t)] + B[\cos(\omega t) - i \sin(\omega t)] \\ &= (A + B) \cos(\omega t) + i(A - B) \sin(\omega t), \end{aligned}$$

and this can be expressed as

$$y(t) = C \cos(\omega t) + D \sin(\omega t), \tag{18}$$

where $C = A + B$ and $D = i(A - B)$ are arbitrary constants. If $y(t)$ is real, then the constants C and D are real-valued. In this case, the constants A and B are not real-valued, but that does not matter. The important thing is that we have obtained the general solution of equation (16) in the form of equation (18) and this, of course, agrees with the solution given in the Introduction.

2.2 Solution in the general case

The method just described works far more generally. Suppose that we wish to solve the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{19}$$

For variety, we use the symbol x for the independent variable.

where a, b and c are constants, with $a \neq 0$. This is the general form of a homogeneous linear second-order equation with constant coefficients.

Then we start by substituting the **trial solution**

$$y = e^{\lambda x},$$

where λ is an undetermined constant, into the differential equation. We have $dy/dx = \lambda e^{\lambda x}$ and $d^2y/dx^2 = \lambda^2 e^{\lambda x}$, so substituting $y = e^{\lambda t}$ into the left-hand side of equation (19) gives

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} \\ &= (a\lambda^2 + b\lambda + c)e^{\lambda x}. \end{aligned}$$

Hence $y = e^{\lambda x}$ is a solution of equation (19) provided that λ satisfies the quadratic equation

$$a\lambda^2 + b\lambda + c = 0. \tag{20}$$

This equation plays such an important role in solving linear constant-coefficient second-order differential equations that it is given a special name.

The auxiliary equation is sometimes called the **characteristic equation**.

Definition

The **auxiliary equation** of the homogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is the quadratic equation

$$a\lambda^2 + b\lambda + c = 0.$$

There is no need to write down all the steps that led to the auxiliary equation. You can just use the rules that emerge from the calculation leading to equation (20): the auxiliary equation is obtained from the differential equation by

$$\text{replacing } \frac{d^2y}{dx^2} \text{ by } \lambda^2, \ \frac{dy}{dx} \text{ by } \lambda, \text{ and } y \text{ by } 1.$$

Example 2

Write down the auxiliary equation of the differential equation

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0.$$

Solution

The auxiliary equation is

$$3\lambda^2 - 2\lambda + 4 = 0.$$

Exercise 2

Write down the auxiliary equation of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ (b) $y'' - 9y = 0$ (c) $\ddot{x} + 2\dot{x} = 0$

We know that $y = e^{\lambda x}$ is a solution of equation (19) provided that λ satisfies the corresponding auxiliary equation. But the auxiliary equation is a quadratic equation, so it has two roots, λ_1 and λ_2 say. For the moment, we assume that these are distinct: $\lambda_1 \neq \lambda_2$. Corresponding to these roots, there are two distinct solutions of differential equation (19):

$$y_1(x) = e^{\lambda_1 x} \quad \text{and} \quad y_2(x) = e^{\lambda_2 x}.$$

We now follow the logic of the preceding subsection. The principle of superposition implies that any linear combination of $y_1(x)$ and $y_2(x)$ satisfies the differential equation. It therefore follows that the function

$$y(x) = C\,y_1(x) + D\,y_2(x), \tag{21}$$

where C and D are arbitrary constants, satisfies the differential equation. This solution contains two arbitrary constants, as expected for the general solution of a second-order differential equation. We therefore conclude that equation (21) is the general solution of equation (19). This important result is summarised as a theorem.

Theorem 1 General solution of homogeneous equations

Given a homogeneous linear second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

with constant coefficients $a \neq 0$, b and c, the auxiliary equation is

$$a\lambda^2 + b\lambda + c = 0.$$

This usually has two distinct roots, λ_1 and λ_2, associated with two distinct solutions $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$. Provided that the roots are distinct, the general solution of the differential equation is

$$y(x) = Ce^{\lambda_1 x} + De^{\lambda_2 x}, \tag{22}$$

where C and D are arbitrary constants.

The roots of the auxiliary equation are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{23}$$

It does not matter which of the roots is called λ_1 and which is called λ_2.

Assuming that the coefficients a, b and c are real, there are three cases to consider, depending on the sign of the discriminant $b^2 - 4ac$:

- For $b^2 - 4ac > 0$, the roots are distinct and real.
- For $b^2 - 4ac < 0$, the roots are distinct and complex.
- For $b^2 - 4ac = 0$, the roots are equal and real.

We consider each of these cases in turn.

Distinct real roots

Example 3

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

and find its roots λ_1 and λ_2.

(b) Use Theorem 1 to write down the general solution of the differential equation, and verify that your answer does satisfy the differential equation.

Solution

(a) The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

This equation may be solved, for example, by factorising it in the form

Using the formula

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

produces the same answer.

$$(\lambda - 1)(\lambda - 2) = 0,$$

to give the two roots $\lambda_1 = 1$ and $\lambda_2 = 2$.

(b) Since $\lambda_1 = 1$ and $\lambda_2 = 2$ are the roots of the auxiliary equation, the functions $y_1 = e^x$ and $y_2 = e^{2x}$ are solutions of the differential equation. Theorem 1 then shows that the general solution of the differential equation is

$$\begin{aligned} y(x) &= C\,y_1(x) + D\,y_2(x) \\ &= Ce^x + De^{2x}, \end{aligned}$$

where C and D are arbitrary constants.

To check that this function satisfies the differential equation, we differentiate it and then differentiate again, to get

$$\frac{dy}{dx} = Ce^x + 2De^{2x} \quad \text{and} \quad \frac{d^2y}{dx^2} = Ce^x + 4De^{2x}.$$

Substituting into the left-hand side of the differential equation then gives

$$\begin{aligned} &\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \\ &\quad = \left(Ce^x + 4De^{2x}\right) - 3\left(Ce^x + 2De^{2x}\right) + 2\left(Ce^x + De^{2x}\right) \\ &\quad = C(1 - 3 + 2)e^x + D(4 - 6 + 2)e^{2x} \\ &\quad = 0, \end{aligned}$$

as required. Hence $y = Ce^x + De^{2x}$ is a solution of the differential equation, for all values of C and D.

Exercise 3

Use the auxiliary equation and Theorem 1 to find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$ (b) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 0$ (c) $\dfrac{d^2z}{du^2} - 4z = 0$

Distinct complex conjugate roots

When the discriminant $b^2 - 4ac$ is negative, equations (23) produce two complex roots that are complex conjugates of one another:

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i,$$

Recall that the complex conjugate of $\alpha + \beta i$ is $\alpha - \beta i$.

where α and β are real. The corresponding functions $y_1(x) = Ae^{\lambda_1 x}$ and $y_2(x) = Be^{\lambda_2 x}$ satisfy the differential equation, and the general solution takes the form

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x}, \tag{24}$$

where A and B are arbitrary constants (which may be complex in this case). We usually need a real-valued solution, so it is best to express our result without using complex numbers. This can be achieved by first writing equation (24) as

$$\begin{aligned} y(x) &= Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \\ &= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\ &= e^{\alpha x}\big(Ae^{i\beta x} + Be^{-i\beta x}\big). \end{aligned}$$

We can simplify this by using Euler's formula. Following the same argument as that which led to equation (18), but with βx in place of ωt, we conclude that

$$y(x) = e^{\alpha x}\big(C\cos(\beta x) + D\sin(\beta x)\big), \tag{25}$$

where $C = A + B$ and $D = i(A - B)$ are arbitrary constants. If the required solution is real-valued, then C and D are real-valued, and equation (25) is the most convenient form of the general solution.

Example 4

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0,$$

and show that its roots are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

(b) Hence write down the general solution of the differential equation in terms of sines and cosines.

Solution

(a) The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0.$$

The standard formula for the roots of a quadratic gives

$$\lambda = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i,$$

so the two complex conjugate roots are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

(b) The roots are $\alpha \pm i\beta$, where $\alpha = 3$ and $\beta = 2$, so using equation (25), the general solution of the differential equation is

$$y(x) = e^{3x}(C \cos 2x + D \sin 2x),$$

where C and D are arbitrary constants.

Exercise 4

Use the auxiliary equation and Theorem 1 to find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0$ (b) $\dfrac{d^2\theta}{dt^2} + 9\theta = 0$

Equal roots

There is a special case where the method based on Theorem 1 does not work. This is when the two roots of the auxiliary equation are *equal*.

To see what the problem is, suppose that we have two solutions $y_1(x)$ and $y_2(x)$ of a homogeneous linear second-order differential equation. Then the principle of superposition tells us that any linear combination

$$y(x) = C\,y_1(x) + D\,y_2(x) \tag{26}$$

is also a solution, and we have argued that this must be the general solution because it contains two arbitrary constants. But suppose that the functions $y_1(x)$ and $y_2(x)$ are constant multiples of one another, so that $y_2(x) = k\,y_1(x)$ for all x, where k is a constant. In this case, we can rewrite equation (26) as

$$y(x) = C\,y_1(x) + Dk\,y_1(x) = (C + kD)\,y_1(x),$$

which shows that there is really only one arbitrary constant, $A = C + kD$, in this case.

If two functions are constant multiples of one another, then they are said to be **linearly dependent**. In order for equation (26) to be the general solution of the homogeneous linear second-order differential equation, the functions y_1 and y_2 must *not* be constant multiples of one another; we say that they must be **linearly independent** solutions.

If both the roots of the auxiliary equation are equal to λ, then the function $y_1(x) = e^{\lambda x}$ provides one solution of the differential equation, but we still need another *linearly independent* solution in order to construct the general solution. Fortunately, there is a simple way of finding this extra solution, illustrated by the following example.

Example 5

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0,$$

and find its roots.

(b) Deduce that $y_1 = e^{-3x}$ is a solution of the differential equation.

(c) By substituting into the differential equation, show that $y_2 = xe^{-3x}$ is also a solution.

(d) Deduce that $y = (C + Dx)e^{-3x}$ is a solution of the differential equation for any values of the constants C and D. Is this the general solution of the differential equation?

Solution

(a) The auxiliary equation is

$$\lambda^2 + 6\lambda + 9 = 0.$$

The left-hand side is the perfect square $(\lambda + 3)^2$, so the auxiliary equation has equal roots $\lambda_1 = \lambda_2 = -3$.

(b) Since $\lambda_1 = -3$ is a root of the auxiliary equation, $y_1 = e^{-3x}$ is a solution of the differential equation.

(c) To show that $y_2 = xe^{-3x}$ is a solution of the differential equation, we differentiate it twice and substitute into the differential equation. Differentiation gives

$$\frac{dy_2}{dx} = e^{-3x} + x(-3e^{-3x}) = (1 - 3x)e^{-3x},$$
$$\frac{d^2y_2}{dx^2} = -3e^{-3x} + (1 - 3x)(-3e^{-3x}) = (-6 + 9x)e^{-3x}.$$

Here we are using the product rule for differentiation.

Substituting these into the left-hand side of the differential equation then gives

$$\begin{aligned}\frac{d^2y_2}{dx^2} + 6\frac{dy_2}{dx} + 9y_2 &= (-6 + 9x)e^{-3x} + 6(1 - 3x)e^{-3x} + 9xe^{-3x} \\ &= (-6 + 6)e^{-3x} + (9 - 18 + 9)xe^{-3x} \\ &= 0.\end{aligned}$$

Hence $y_2 = xe^{-3x}$ is a solution of the differential equation.

(d) Since $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$ are both solutions of the same homogeneous differential equation, the principle of superposition tells us that

$$y = Ce^{-3x} + Dxe^{-3x} = (C + Dx)e^{-3x}$$

is also a solution of this equation for any values of C and D.

This solution contains two arbitrary constants C and D that cannot be combined because the functions e^{-3x} and xe^{-3x} are linearly independent (i.e. they are not constant multiples of one another). It is therefore the *general solution* of the differential equation.

If you want to prove this, you can substitute $y = xe^{\lambda x}$ into the left-hand side of equation (19), and use the fact that λ satisfies the auxiliary equation, with $\lambda = -b/2a$ for equal roots.

The method used in the above example always works when the auxiliary equation has equal roots. (The proof follows the same method as the example, but using symbols instead of numbers.) Thus whenever the auxiliary equation has equal roots $\lambda_1 = \lambda_2$, the *general solution* of the homogeneous equation is

$$y(x) = (C + Dx)e^{\lambda_1 x}, \tag{27}$$

where C and D are arbitrary constants.

Exercise 5

Use the auxiliary equation method to find the general solution of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ (b) $\ddot{s} - 4\dot{s} + 4s = 0$

2.3 General procedure and further practice

We have now considered all the cases that can arise when solving a homogeneous linear second-order differential equation with constant coefficients. This subsection summarises the method of solution as a procedure, and gives exercises for further practice.

Procedure 1 General solution of a homogeneous linear constant-coefficient second-order differential equation

The general solution of the homogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{Eq. 19}$$

where a, b, c are real constants with $a \neq 0$, may be found as follows.

1. Write down the auxiliary equation

$$a\lambda^2 + b\lambda + c = 0, \tag{Eq. 20}$$

and find its roots λ_1 and λ_2.

2. (a) If the auxiliary equation has two distinct real roots λ_1 and λ_2, then the general solution is

$$y = Ce^{\lambda_1 x} + De^{\lambda_2 x}. \qquad \text{(Eq. 22)}$$

(b) If the auxiliary equation has a pair of complex conjugate roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, then the general solution is

$$y = e^{\alpha x}(C\cos\beta x + D\sin\beta x). \qquad \text{(Eq. 25)}$$

(c) If the auxiliary equation has two equal real roots $\lambda_1 = \lambda_2$, then the general solution is

$$y = (C + Dx)e^{\lambda_1 x}. \qquad \text{(Eq. 27)}$$

In each of these cases, C and D are arbitrary constants.

Exercise 6

Use Procedure 1 to find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 4y = 0$ (b) $\dfrac{d^2y}{dx^2} - 9y = 0$

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$ (d) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$

(e) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 29y = 0$ (f) $u''(x) - 6u'(x) + 8u(x) = 0$

Exercise 7

(a) Write down the auxiliary equation of the differential equation

$$3\frac{dy}{dx} - y - 2\frac{d^2y}{dx^2} = 0.$$

(b) Solve this auxiliary equation, and write down the general solution of the differential equation.

Exercise 8

Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$ (b) $\dfrac{d^2y}{dx^2} - 16y = 0$

(c) $\dfrac{d^2y}{dx^2} + 4y = 4\dfrac{dy}{dx}$ (d) $\dfrac{d^2\theta}{dt^2} + 3\dfrac{d\theta}{dt} = 0$

Exercise 9

For which values of the constant k does the differential equation

$$\frac{d^2y}{dx^2} + 4k\frac{dy}{dx} + 4y = 0$$

have a general solution with oscillating behaviour, that is, a general solution which involves sines and cosines?

2.4 Damped harmonic oscillators

Apart from equations (31) and (32), and the definitions of *amplitude*, *phase constant*, *angular frequency* and *period*, this subsection contains no new mathematical ideas, so if you are short of time, you may choose to read it quickly. However, many students find it invaluable to think about the solutions of differential equations in the context of real oscillating systems.

We consider again the *damped harmonic oscillator* that was discussed in the Introduction. This system was illustrated in Figure 1. It consists of an object of mass m that moves along the x-axis, with position $x(t)$ at time t. A spring exerts a force $-kx$ on the object, pulling it towards the equilibrium position $x = 0$. The object also experiences a damping (or frictional) force that is taken to be proportional to the object's velocity dx/dt. Under these circumstances, the object obeys the equation of motion given in equation (6). This is the homogeneous linear constant-coefficient differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0, \tag{28}$$

where m, k and γ are positive constants. The damped harmonic oscillator is a good model for vehicle suspension systems, but equations like (28) appear in many other contexts where oscillations occur.

Let us begin with the special case of no damping ($\gamma = 0$). Equation (28) then reduces to the equation of motion of a simple harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{k/m}$. You saw earlier that the general solution of this equation is

$$x(t) = C\cos(\omega t) + D\sin(\omega t), \tag{Eq. 18}$$

where C and D are arbitrary constants.

To interpret this solution, it is helpful to note that it can also be written as

$$x(t) = A\sin(\omega t + \phi), \tag{29}$$

where A and ϕ are arbitrary constants. To see why this works, expand the right-hand side of equation (29), to get

$$x(t) = A\sin(\omega t)\cos\phi + A\cos(\omega t)\sin\phi.$$

Recall the trigonometric identity
$\sin(A+B)$
$= \sin A\cos B + \cos A\sin B.$

Then comparing with equation (18), we see that equation (29) is valid provided that

$$C = A\sin\phi \quad \text{and} \quad D = A\cos\phi. \tag{30}$$

Squaring and adding these equations gives

$$C^2 + D^2 = A^2(\sin^2\phi + \cos^2\phi) = A^2,$$

so

$$A = \sqrt{C^2 + D^2}. \tag{31}$$

Dividing the first equation in (30) by the second, we also have

$$\frac{C}{D} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi,$$

so

$$\phi = \arctan(C/D). \tag{32}$$

In equation (31) we have chosen the positive square root for A; this involves no loss of generality because $\sin(\omega t + \pi) = -\sin(\omega t)$, so increasing the value of ϕ by π is equivalent to reversing the sign of A. Values of ϕ that differ by an integer multiple of 2π correspond to the same motion, so we can restrict ϕ to a range such as $0 \le \phi < 2\pi$.

Figure 4 shows a graph of the solution.

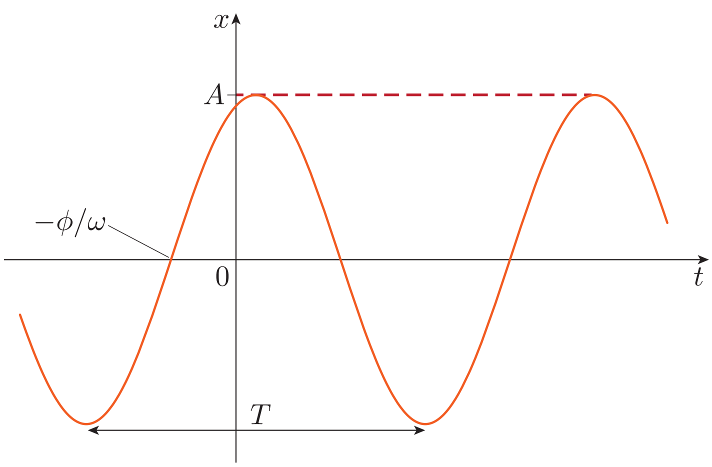

Figure 4 Simple harmonic motion with amplitude A, phase constant ϕ and period T

The constant $A \geq 0$ is the **amplitude** of the oscillation, which is the magnitude of the maximum displacement from the equilibrium position $x = 0$. The constant ϕ is called the **phase constant**. This is related to the time when the oscillator passes through the equilibrium position: according to equation (29), $x = 0$ at time $t = -\phi/\omega$. The oscillation consists of identical cycles, endlessly repeated. The time taken to complete one of these cycles is the **period** of the oscillation, and is given the symbol T. Because the sine function has period 2π, we have

$$\omega T = 2\pi$$

and so

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

As you might expect, the period is reduced if m is reduced (a lighter particle) or if k is increased (a stiffer spring). The quantity $1/T$ is called the **frequency** of the oscillation, and represents the number of cycles completed per unit time. The constant $\omega = 2\pi/T$ is called the **angular frequency**.

ω is sometimes called the **natural angular frequency**.

Now let us return to the case where damping is present ($\gamma > 0$). In this case, the relevant homogeneous differential equation is (28), which can be written more simply as

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega^2 x = 0, \tag{33}$$

where the **damping parameter** $\Gamma = \gamma/(2m)$ is a measure of the damping, and $\omega = \sqrt{k/m}$ is the angular frequency of the corresponding *undamped* oscillator. The corresponding auxiliary equation is

$$\lambda^2 + 2\Gamma\lambda + \omega^2 = 0, \tag{34}$$

and this has solutions

$$\lambda = -\Gamma \pm \sqrt{\Gamma^2 - \omega^2}. \tag{35}$$

Depending on the value of the discriminant $\Gamma^2 - \omega^2$, there are three different types of solution, which correspond to three different types of motion. These exemplify the three cases described in Subsection 2.2: distinct complex roots, distinct real roots and equal roots.

Underdamped motion: $\Gamma < \omega$

When $\Gamma < \omega$, the discriminant $\Gamma^2 - \omega^2$ is negative, and the roots of the auxiliary equation are complex numbers:

$$\lambda = -\Gamma \pm i\Omega, \quad \text{where } \Omega = \sqrt{\omega^2 - \Gamma^2}. \tag{36}$$

The general solution is therefore

$$x(t) = e^{-\Gamma t}\left[C\cos(\Omega t) + D\sin(\Omega t)\right], \tag{37}$$

where C and D are arbitrary constants. Transforming the term in square brackets, just as in equation (29), we can also write this as

$$x(t) = Ae^{-\Gamma t}\sin(\Omega t + \phi). \tag{38}$$

Figure 5 shows graphs of this function for different values of the arbitrary constants A and ϕ. (Section 4 will explain how these arbitrary constants can be determined from given initial conditions.)

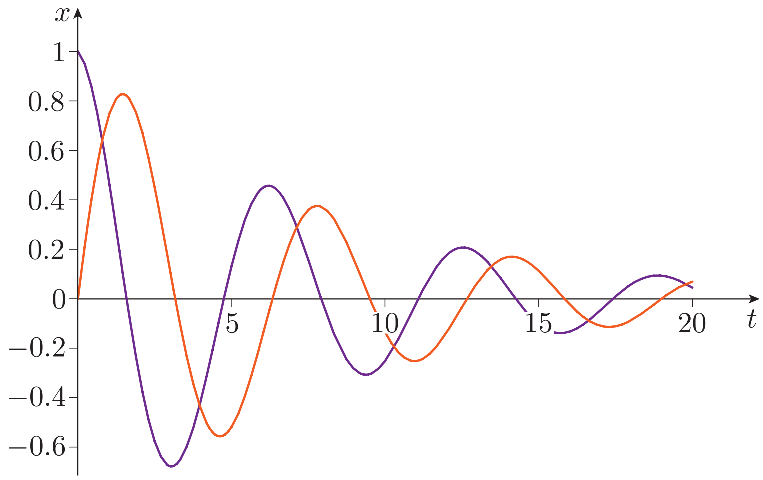

Figure 5 Two solutions for an underdamped harmonic oscillator with $\omega = 1$ and $\Gamma = 1/8$, corresponding to $A = 1$, $\phi = 0$ (orange curve) and $A = 1$, $\phi = \pi/2$ (purple curve)

This motion is called **underdamped** because the damping force is weak enough for the motion of the particle to oscillate to and fro. The angular frequency Ω of the damped oscillator is smaller than the angular frequency ω of the corresponding undamped oscillator (see equation (36)). Each cycle of the damped oscillator takes a period $T = 2\pi/\Omega$, which is longer than the period $2\pi/\omega$ of the corresponding simple harmonic oscillation, although this effect is slight if $\Gamma \ll \omega$. The amplitude of the damped oscillations is $Ae^{-\Gamma t}$, which decreases exponentially with time. In this context, Γ may be called the **decay constant**. If Γ approaches zero, the decay constant approaches zero and the angular frequency Ω approaches ω, so in this limit the system behaves like a simple harmonic oscillator.

Overdamped motion: $\Gamma > \omega$

When $\Gamma > \omega$, the discriminant $\Gamma^2 - \omega^2$ is positive, and the roots of the auxiliary equation are real negative numbers:

$$\lambda_1 = -\Gamma - \sqrt{\Gamma^2 - \omega^2} \quad \text{and} \quad \lambda_2 = -\Gamma + \sqrt{\Gamma^2 - \omega^2}.$$

In this case, the general solution of the homogeneous differential equation is

$$x(t) = Ce^{\lambda_1 t} + De^{-\lambda_2 t}, \qquad (39)$$

where C and D are arbitrary constants. Figure 6 shows graphs of this function for different values of the arbitrary constants C and D.

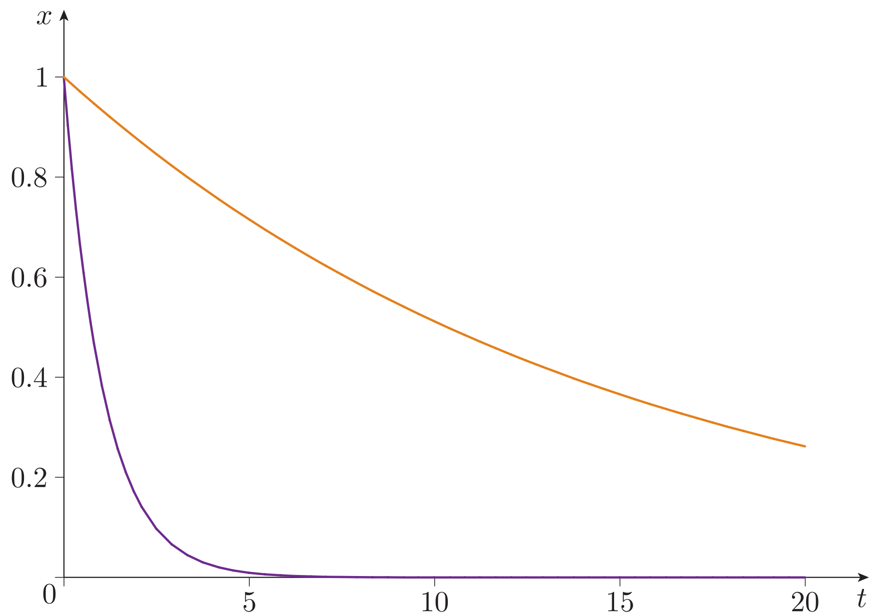

Figure 6 Two solutions for an overdamped harmonic oscillator with $\omega = 1/4$ and $\Gamma = 1/2$, corresponding to $C = 1$, $D = 0$ (orange curve) and $C = 0$, $D = 1$ (purple curve)

Because λ_1 and λ_2 are both negative, the solution dies away exponentially, and there are no oscillations. This motion is called **overdamped** because the damping force is strong enough to prevent oscillations. When Γ approaches ω from above, both roots approach the value Γ, and the solution is proportional to $e^{-\Gamma t}$.

Critically damped motion: $\Gamma = \omega$

When $\Gamma = \omega$, the discriminant $\Gamma^2 - \omega^2$ is equal to zero. In this case the general solution is

$$x(t) = (C + Dt)e^{-\Gamma t}, \qquad (40)$$

where C and D are arbitrary constants. Figure 7 shows graphs of this function for different values of the arbitrary constants C and D.

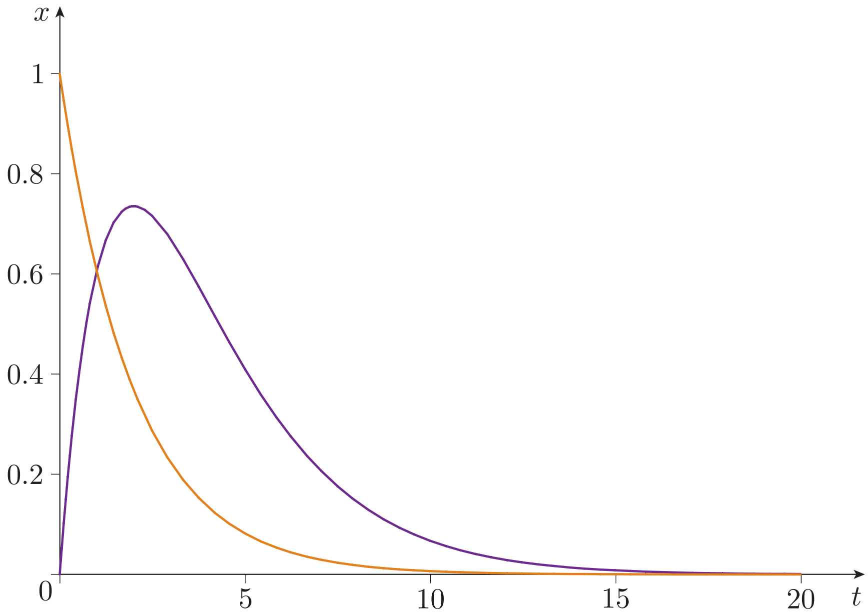

Figure 7 Two solutions for a critically damped harmonic oscillator with $\omega = \Gamma = 1/2$, corresponding to $C = 1$, $D = 0$ (orange curve) and $C = 0$, $D = 1$ (purple curve)

The motion is described as **critically damped**. A vehicle suspension system is usually set up to be close to critical damping, so that it is soft enough to move in response to a bumpy road, without allowing oscillations.

Exercise 10

Small oscillations of the pendulum shown in Figure 8 can be described by the homogeneous differential equation

$$m\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \frac{mg}{l}\theta = 0,$$

where m is the mass of the bob, l is the length of the string, g is the magnitude of the acceleration due to gravity, and γ is a damping constant. With mass measured in kilograms, length in metres and time in seconds, the values of these constants are $m = 0.80$, $l = 2.0$, $g = 9.8$ and $\gamma = 0.016$.

(a) Is this oscillation underdamped, overdamped or critically damped?

(b) What is the period of the oscillation?

(c) If the initial amplitude of the oscillation is $\theta = 0.20$ (in radians), what is the amplitude at $t = 100$?

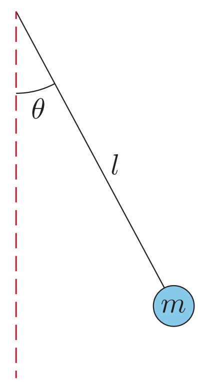

Figure 8 A pendulum

Exercise 11

A critically damped harmonic oscillator is described by the differential equation (28), with $m = 1$ and $k = 4$ (in suitable units). Determine the value of Γ in these units, and write down an expression for the general solution $x(t)$.

3 Inhomogeneous differential equations

3.1 General method of solution

Section 2 focused on finding the general solution of *homogeneous* linear constant-coefficient second-order differential equations. This section explains how to find the general solution of *inhomogeneous* linear constant-coefficient second-order differential equations – that is, equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where a, b, c are constants, with $a \neq 0$, and $f(x)$ is a given continuous function of x. The basic method for finding the general solution of such an equation depends on the principle of superposition, and is illustrated in the following example.

Example 6

Exercise 3(a) showed that the homogeneous equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \tag{41}$$

The notation $y_c(x)$ and $y_p(x)$ (see below) will be explained shortly.

has the general solution $y_c(x) = Ce^{-2x} + De^{-3x}$, where C and D are arbitrary constants.

Use this fact, together with the principle of superposition, to show that the inhomogeneous equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12 \tag{42}$$

has a solution $y(x) = Ce^{-2x} + De^{-3x} + 2$ for any C and D.

Solution

We can show that the constant function $y_p(x) = 2$ is a particular solution of the inhomogeneous equation by substituting it into the left-hand side. This gives

$$\frac{d^2y_p}{dx^2} + 5\frac{dy_p}{dx} + 6y_p = 0 + 5 \times 0 + 6 \times 2 = 12,$$

which is the same as the right-hand side, as required.

We know that $y_c(x) = Ce^{-2x} + De^{-3x}$ is a solution of equation (41) for any C and D, and $y_p(x) = 2$ is a solution of equation (42). The principle of superposition then tells us that $y_c(x) + y_p(x)$ is a solution of

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12 + 0 = 12,$$

which is identical to equation (42). Hence $y(x) = Ce^{-2x} + De^{-3x} + 2$ is a solution of equation (42) for any C and D.

The solution found in the example above contains two arbitrary constants, C and D (which appear as coefficients of two linearly independent functions, e^{-2x} and e^{-3x}). We expect the general solution of any second-order differential equation to contain two independent arbitrary constants, so $y(x) = Ce^{-2x} + De^{-3x} + 2$ is the general solution of equation (42). We will now generalise this idea, but first it is helpful to introduce some terminology.

Corresponding to the inhomogeneous equation (42), we have the homogeneous equation (41), obtained by replacing the function on the right-hand side by zero. This is called the *associated homogeneous equation*. The solutions y_{c} and y_{p} also have special names in this context: $y_{\mathrm{c}}(x)$, the general solution of the associated homogeneous equation (41), is called the *complementary function*, and $y_{\mathrm{p}}(x)$, a particular solution of the inhomogeneous equation (42), is called a *particular integral*.

Definitions

Let

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{43}$$

be an inhomogeneous linear constant-coefficient second-order differential equation.

- Its **associated homogeneous equation** is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{44}$$

- The general solution $y_{\mathrm{c}}(x)$ of the associated homogeneous equation is known as the **complementary function** for the original inhomogeneous equation (43).
- Any particular solution $y_{\mathrm{p}}(x)$ of the original inhomogeneous equation (43) is referred to as a **particular integral** for that equation.

The term *particular integral* is used here, rather than the term particular solution, which we reserve for a solution that contains definite numbers rather than arbitrary constants (see Section 4).

The particular integral is not unique – many different choices can be made, but that does not matter. Suppose that y_{p_1} and y_{p_2} are two different particular integrals for equation (43). Then the principle of superposition tells us that $y_{\mathrm{p}_2} - y_{\mathrm{p}_1}$ is a solution of the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) - f(x) = 0,$$

which is the associated homogeneous equation (44). But we know that the general solution of this equation is given by the complementary function y_{c}, so it follows that $y_{\mathrm{p}_2} - y_{\mathrm{p}_1} = y_{\mathrm{c}}$, or equivalently,

$$y_{\mathrm{p}_2} = y_{\mathrm{c}} + y_{\mathrm{p}_1}.$$

Since any solution can be written in this form, it follows that $y_c + y_{p_1}$ is the *general solution* of the inhomogeneous equation (43). We have therefore proved the following important result.

Theorem 2 General solution of an inhomogeneous equation

The **general solution** of a linear constant-coefficient second-order differential equation is given by

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is the complementary function (the general solution of the associated homogeneous equation) and $y_p(x)$ is a particular integral of the inhomogeneous equation.

Note that $y_c(x)$, being the general solution of the associated homogeneous equation, will contain *two* arbitrary constants, whereas $y_p(x)$, being a particular solution of equation (43), will contain none. You have already seen how to find the complementary function $y_c(x)$ in Subsection 2.2. There is no general recipe for finding a particular integral $y_p(x)$, but there are methods for 'guessing' a suitable solution, which work in most cases. The following example illustrates the general technique.

Example 7

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 9x + 9. \qquad (45)$$

Solution

The associated homogeneous equation is

$$\frac{d^2y}{dx^2} + 9y = 0,$$

which has the general solution

$$y_c = C\cos 3x + D\sin 3x,$$

See Exercise 4(b), although there different symbols are used for the variables.

where C and D are arbitrary constants. This is the complementary function for equation (45).

You will see in the next subsection how to find such a particular integral.

A particular integral for equation (45) is

$$y_p = x + 1.$$

This may be verified by differentiation and substitution: $y_p'' = 0$, so substituting into the left-hand side of equation (45) gives

$$y_p'' + 9y_p = 0 + 9(x + 1) = 9x + 9,$$

which is the same as the right-hand side of equation (45), as required.

The general solution of equation (45) is therefore, by Theorem 2,

$$y = y_{\mathrm{c}} + y_{\mathrm{p}} = C\cos 3x + D\sin 3x + x + 1,$$

where C and D are arbitrary constants.

The method of Example 7 may be summarised as follows.

Procedure 2 General solution of an inhomogeneous linear constant-coefficient second-order differential equation

The **general solution** of the inhomogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

is found as follows.

1. First find the complementary function $y_{\mathrm{c}}(x)$, i.e. the general solution of the associated homogeneous equation

   $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

   using Procedure 1.
2. Then find a particular integral $y_{\mathrm{p}}(x)$.
3. The general solution is then $y(x) = y_{\mathrm{c}}(x) + y_{\mathrm{p}}(x)$.

The reason why y_{c} is found first will become clear in Subsection 3.3.

It is worth noting that, by Theorem 2, *any* choice of particular integral in Procedure 2 gives the *same* general solution. Formulas obtained for the general solution may look different for different choices of particular integral, but they are in fact always equivalent. For example, in Example 7 the particular integral $y_{\mathrm{p}} = x + 1$ was chosen, and the general solution was obtained as $y = C\cos 3x + D\sin 3x + x + 1$. It would have been equally valid to have chosen $y_{\mathrm{p}} = x + 1 + \sin 3x$ as the particular integral. In that case, the general solution would have been obtained as $y = C\cos 3x + D\sin 3x + x + 1 + \sin 3x$. This form looks a little different, but it may be written as $y = C\cos 3x + (D+1)\sin 3x + x + 1$; and since C and D are arbitrary constants, this form of the general solution represents exactly the same family of solutions.

Exercise 12

For each of the following differential equations:

- Write down its associated homogeneous equation and its complementary function y_{c}.
- Find a particular integral of the form $y_{\mathrm{p}} = p$, where p is a constant.
- Write down the general solution of the differential equation.

(a) $\dfrac{d^2y}{dx^2} + 4y = 8$ (b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 6$

The complementary functions can be found from Exercise 6(a) and Example 3.

In Exercise 12, where the right-hand sides of the equations are constants, it is possible to find a particular integral almost 'by inspection'; but this method is generally inadequate. Fortunately, there exist procedures for finding a particular integral for equations involving wide classes of right-hand-side functions $f(x)$. The remainder of this section considers some of the simpler cases, where it is possible to determine the *general form* of a particular integral by inspection, although some manipulation is needed to determine the values of certain coefficients.

3.2 The method of undetermined coefficients

You have seen that the linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad \text{(Eq. 43)}$$

has the general solution $y(x) = y_{\mathrm{c}}(x) + y_{\mathrm{p}}(x)$, where the complementary function $y_{\mathrm{c}}(x)$ is found by solving the associated homogeneous equation, using Procedure 1. However, the methods for finding a particular integral $y_{\mathrm{p}}(x)$ are another matter.

We can proceed by guessing that $y_{\mathrm{p}}(x)$ takes some general form, called a **trial solution**, which involves one or more constants (or coefficients) whose values are initially undetermined. Then, by substituting the trial solution into the differential equation, we can hope to find the values of these coefficients, and hence find a particular integral. This is called the **method of undetermined coefficients**.

You saw an example of this method in Exercise 12. There the function $f(x)$ on the right-hand side of the differential equation is a *constant*, and the trial solution is taken to be of the form $y_{\mathrm{p}} = p$, where p is an unknown constant, whose value is determined by substituting into the differential equation.

The choice of trial solution depends on the function $f(x)$ on the right-hand side of equation (43). We will look at three cases:

- polynomial functions
- functions with exponential behaviour
- sinusoidal functions.

When we have considered examples for each case, we will gather everything together as a general procedure. Bear in mind, though, that the method finds only a particular integral for the differential equation; to find the *general solution* you also need to find the complementary function and add this to the particular integral, according to Procedure 2.

Polynomial functions

If $f(x)$ is a given polynomial, we have

$$f(x) = m_n x^n + m_{n-1} x^{n-1} + \cdots + m_1 x + m_0,$$

where $m_0, m_1, \ldots, m_n$ are given constants. This class of functions includes constant functions ($n = 0$), linear functions ($n = 1$), quadratic functions ($n = 2$) and higher-order polynomials ($n \geq 3$).

Let us start by considering the case where $f(x)$ is a linear function (a polynomial of degree 1).

Example 8

Find a particular integral for

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4x + 2.$$

Solution

We try a solution of the form

$$y = p_1 x + p_0,$$

For simplicity, we denote the trial solution by y, with no subscript p.

where p_1 and p_0 are coefficients to be determined so that the differential equation is satisfied. To try this solution, we need the first and second derivatives of y:

$$\frac{dy}{dx} = p_1, \quad \frac{d^2y}{dx^2} = 0.$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned} 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= 3 \times 0 - 2p_1 + (p_1 x + p_0) \\ &= p_1 x + (p_0 - 2p_1). \end{aligned}$$

For $y = p_1 x + p_0$ to be a solution of the differential equation, we require that

$$p_1 x + (p_0 - 2p_1) = 4x + 2 \quad \text{for all } x. \tag{46}$$

To find the two unknown coefficients p_1 and p_0, we compare the coefficients on both sides of equation (46). Comparing the terms in x gives $p_1 = 4$. Comparing the constant terms gives $p_0 - 2p_1 = 2$, so $p_0 = 2 + 2p_1 = 2 + 2 \times 4 = 10$. Therefore we have the particular integral

Notice that we can get two separate bits of information from the same equation because it applies for all x.

$$y_{\mathrm{p}} = 4x + 10.$$

Check: If $y_{\mathrm{p}} = 4x + 10$, then $dy_{\mathrm{p}}/dx = 4$, $d^2y_{\mathrm{p}}/dx^2 = 0$, and substituting into the left-hand side of the differential equation gives

$$3\frac{d^2y_{\mathrm{p}}}{dx^2} - 2\frac{dy_{\mathrm{p}}}{dx} + y_{\mathrm{p}} = 3 \times 0 - 2 \times 4 + (4x + 10) = 4x + 2,$$

as required.

In the example above, the target function $f(x) = 4x + 2$ was a linear function, and the trial solution $p_1x + p_0$ was also a linear function. When this trial solution was substituted into the left-hand side of the differential equation, it produced another linear function, $p_1x + (p_0 - 2p_1)$, whose coefficients could be compared with those of the target function.

This is really the key to the method. The idea is to choose a trial solution that includes undetermined constants which, when substituted into the left-hand side of the differential equation, generates a function that can be compared directly with the target function $f(x)$, allowing the constants to be found.

This generally means that the trial solution should be chosen to belong to the same class of functions as $f(x)$ on the right-hand side of the equation. However, we must choose a trial solution that is general enough, so that its first and second derivatives also belong to the same class as the trial solution itself. This is illustrated in the following exercise.

Exercise 13

Use trial solutions of the form $y = p_1x + p_0$ to find particular integrals for each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 2x + 3$ (b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x$

Note that the trial solution $y = p_1x + p_0$ was suggested in Exercise 13(b), even though $f(x)$ is just a multiple of x and contains no constant term. In fact, a trial solution of the form p_1x would not work in this case. The reason is that we need to take the first and second derivatives of the trial solution and substitute them into the differential equation. These derivatives must belong to the class encompassed by the trial solution. In this case, the first derivative of p_1x is the constant function p_1, so the trial solution must contain a constant term.

In general, the following advice can be given:

You saw examples of this in Exercise 12.

- If $f(x) = m_0$ is a constant function, then you should use a constant trial solution of the form $y = p_0$.
- If $f(x) = m_1x + m_0$ is a linear function, then you should use a linear trial solution of the form $y = p_1x + p_0$. Even if $m_0 = 0$, you should not initially assume that $p_0 = 0$.
- More generally, if $f(x) = m_nx^n + m_{n-1}x^{n-1} + \cdots + m_1x + m_0$, where $m_n \neq 0$, then you should use a trial solution of the form $y = p_nx^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0$. Even if some of the coefficients $m_0, m_1, \ldots, m_{n-1}$ are equal to zero, you should not initially assume that any of the coefficients $p_0, p_1, \ldots, p_n$ are equal to zero.

Exercise 14

Find a particular integral for

$$y'' - y = t^2.$$

Functions with exponential behaviour

Now let us suppose that the target function $f(x)$ on the right-hand side of the inhomogeneous equation takes the form $f(x) = me^{kx}$, where m and k are constants. In general, such functions are not *the* exponential function, but we can say that they exhibit **exponential behaviour**. The following example shows how to find a particular integral in such a case.

Example 9

Find a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x}.$$

Solution

We try a solution of the form

$$y = pe^{3x},$$

Since the derivative of e^{3x} is $3e^{3x}$, the exponent $(3x)$ appearing in $y(x)$ should be the same as that appearing in $f(x)$, then only the coefficient p is to be determined.

where p is an undetermined coefficient that can be found by requiring that the differential equation is satisfied. Differentiating $y = pe^{3x}$ gives

$$\frac{dy}{dx} = 3pe^{3x}, \quad \frac{d^2y}{dx^2} = 9pe^{3x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} + 9y = 9pe^{3x} + 9pe^{3x} = 18pe^{3x}.$$

Therefore, for $y = pe^{3x}$ to be a solution of the differential equation, we require that $18pe^{3x} = 2e^{3x}$ for all x. Hence $p = \frac{1}{9}$, and

$$y_{\text{p}} = \tfrac{1}{9}e^{3x}$$

is a particular integral for the given differential equation.

The general rule is as follows: when $f(x) = me^{kx}$, we use a trial solution of the form $y(x) = pe^{kx}$, where p is an undetermined constant. The value of k in $y(x)$ is the same as in $f(x)$.

Exercise 15

Find a particular integral for

$$2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^{-x}.$$

Sinusoidal functions

Finally, let us suppose that the target function $f(x)$ on the right-hand side of the inhomogeneous equation takes the form

$$f(x) = m\cos kx + n\sin kx,$$

where m, n and k are constants. Any such function is said to be **sinusoidal**. In this case, the appropriate trial solution is

$$y(x) = p\cos kx + q\sin kx,$$

where p and q are undetermined constants. Following earlier ideas, the trial solution must be general enough to be in the same class as its first and second derivatives. So even if $f(x)$ contains only a sine or only a cosine, the trial solution $y(x)$ must contain both a sine and a cosine. However, the value of the constant k in $y(x)$ should always be the same as that in $f(x)$.

Example 10

Find a particular integral for

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 10\sin 2x.$$

Solution

We try a solution of the form

$$y = p\cos 2x + q\sin 2x,$$

where p and q are coefficients whose values are to be found by substituting into the differential equation. Differentiating y gives

$$\frac{dy}{dx} = -2p\sin 2x + 2q\cos 2x, \quad \frac{d^2y}{dx^2} = -4p\cos 2x - 4q\sin 2x.$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y &= (-4p\cos 2x - 4q\sin 2x) + 2(-2p\sin 2x + 2q\cos 2x)\\ &\qquad + 2(p\cos 2x + q\sin 2x)\\ &= (-2p + 4q)\cos 2x + (-4p - 2q)\sin 2x.\end{aligned}$$

This can be equated to the right-hand side of the differential equation, so

$$(-2p + 4q)\cos 2x + (-4p - 2q)\sin 2x = 10\sin 2x \quad \text{for all } x. \tag{47}$$

Comparing coefficients works because the cos and sin functions are linearly independent.

To find p and q, we compare the coefficients of cos and sin on both sides of equation (47). For this equation to be true for all x, we must have

$$-2p + 4q = 0 \quad \text{and} \quad -4p - 2q = 10.$$

Solving these simultaneous equations, we conclude that $p = -2$, $q = -1$, so

$$y_{\text{p}}(x) = -2\cos 2x - \sin 2x$$

is a particular integral for the given differential equation.

Exercise 16

Find a particular integral for

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = \cos 3t + \sin 3t.$$

The following procedure summarises the results of this subsection.

Procedure 3 Method of undetermined coefficients

To find a **particular integral** for the inhomogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

use a trial solution $y(x)$ with a form similar to that of $f(x)$. The following table gives appropriate trial solutions for simple cases.

There are exceptional cases where these trial solutions do not work; see Subsection 3.3.

Target function $f(x)$	Trial solution $y(x)$
$m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x + m_0$	$p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$
me^{kx}	pe^{kx}
$m\cos kx + n\sin kx$	$p\cos kx + q\sin kx$

The full trial solutions in the right-hand column should be used even when some of the coefficients in $f(x)$ are missing (as in m_2x^2 or $m\cos kx$, for example).

To determine the coefficients in $y(x)$, differentiate it twice, substitute into the left-hand side of the differential equation, and equate coefficients of corresponding terms.

Exercise 17

What form of trial solution y should you use in order to find a particular integral for each of the following differential equations?

In this question, you need not *find* the particular integrals.

(a) $\dfrac{d^2y}{dx^2} - y = e^{3x}$ (b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = \sin 3x$

To round off this discussion of inhomogeneous differential equations, the following exercise gives a variety of equations for further practice.

Exercise 18

Find the general solutions of the following differential equations.

See Exercise 6(a) and Exercise 3(c).

The complementary functions of the differential equations in parts (a) and (c) are $C\cos 2t + D\sin 2t$ and $Ce^{-2x} + De^{2x}$, respectively.

(a) $\dfrac{d^2\theta}{dt^2} + 4\theta = 2t$

(b) $\dfrac{d^2y}{dx^2} + 4y = 10\sin 3x$

(c) $\dfrac{d^2y}{dx^2} - 4y = 15e^{-x}$

3.3 Exceptional cases

There are some exceptional cases for which Procedure 3 fails. You will not encounter these exceptions in assignments or in the exam, but it is useful to have some idea of what can go wrong, and how a particular integral can be found under such circumstances.

To take a definite case, consider the differential equation

$$\frac{d^2y}{dx^2} - 4y = 2e^{2x}. \tag{48}$$

The associated homogeneous equation is

$$\frac{d^2y}{dx^2} - 4y = 0,$$

See Exercise 3(c).

and this has general solution $y = Ce^{-2x} + De^{2x}$. The difficulty is now apparent. Procedure 3 suggests that we use the trial solution $y = pe^{2x}$, but this happens to be a solution of the associated homogeneous equation (with $C = 0$, $D = p$), so substituting $y = pe^{2x}$ into the left-hand side of equation (48) gives zero for any value of p, and this cannot be equal to the non-zero right-hand side.

Difficulties like this are generally overcome by multiplying whichever trial solution is suggested in Procedure 3 by the independent variable, x. So the trial solution to use for equation (48) would be

$$y = pxe^{2x}.$$

Calculating the derivatives of this function, we get

$$\frac{dy}{dx} = pe^{2x} + 2pxe^{2x} = p(1+2x)e^{2x},$$
$$\frac{d^2y}{dx^2} = 2pe^{2x} + 2p(1+2x)e^{2x} = 4p(1+x)e^{2x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} - 4y = 4p(1+x)e^{2x} - 4pxe^{2x} = 4pe^{2x}.$$

Therefore $y = pxe^{2x}$ is a solution of the differential equation provided that $4pe^{2x} = 2e^{2x}$ for all x. Hence $p = \frac{1}{2}$, and

$$y_{\text{p}} = \tfrac{1}{2}xe^{2x}$$

is a particular integral for differential equation (48).

A similar technique works when the inhomogeneous term is a constant, as illustrated in the following example.

Example 11

The motion of a small ball bearing dropped into viscous oil can be modelled by the differential equation

$$m\ddot{x} + r\dot{x} - mg = 0,$$

where m is the mass of the ball, r is a constant related to the viscosity of the oil, g is the magnitude of the acceleration due to gravity, and x is the vertical distance from the point of release.

(a) Find the general solution $x(t)$ of this differential equation.

(b) Use your answer to part (a) to show that the velocity of the ball approaches the limiting value mg/r as t becomes very large.

Solution

(a) The inhomogeneous equation is

$$m\ddot{x} + r\dot{x} = mg,$$

and the auxiliary equation for the associated homogeneous equation is

$$m\lambda^2 + r\lambda = 0.$$

This has solutions $\lambda = 0$ and $\lambda = -r/m$, so the complementary function is

$$x_{\text{c}} = C + De^{-rt/m}.$$

The inhomogeneous term is the constant function mg, so Procedure 3 suggests a trial solution $x = p_0$. However, this is a solution of the associated homogeneous equation (with $C = p_0$, $D = 0$). Hence we try $x = p_0 t$ instead. Differentiating and substituting gives

$$rp_0 = mg,$$

so

$$p_0 = \frac{mg}{r}.$$

Hence a particular integral is

$$x_{\text{p}} = \frac{mgt}{r},$$

and the general solution is

$$x(t) = C + De^{-rt/m} + \frac{mgt}{r},$$

where C and D are arbitrary constants.

(b) The velocity is

$$\dot{x} = -\frac{Dr}{m}\,e^{-rt/m} + \frac{mg}{r}.$$

This approaches the limiting value mg/r as $t \to \infty$.

This answer can also be obtained directly from the differential equation. If the velocity tends towards a constant value, the acceleration $\ddot{x}$ approaches zero, and the differential equation itself tells us that $\dot{x}$ tends to mg/r.

3.4 Combined cases

Another situation that crops up occasionally is when the inhomogeneous term is a linear combination of polynomial, sinusoidal and exponential functions. You know what to do when the inhomogeneous term is $x+1$, and you also know what to do if it is e^{2x}, but what if the inhomogeneous term is $2e^{2x} + 18(x+1)$? The secret is to use the principle of superposition to split the problem into smaller tasks. Again, we include this topic for interest and completeness: assignments and the exam will not contain questions on such combined cases.

Example 12

Find a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x} + 18x + 18. \tag{49}$$

Solution

In Example 9 (Subsection 3.2), you saw that $y_{\text{p}} = \frac{1}{9}e^{3x}$ is a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x}.$$

In Example 7 (Subsection 3.1), you saw that $y_{\text{p}} = x+1$ is a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 9x + 9.$$

Therefore, by the principle of superposition, a particular integral for equation (49) is

$$y_{\text{p}} = \tfrac{1}{9}e^{3x} + 2(x+1) = \tfrac{1}{9}e^{3x} + 2x + 2.$$

This approach can be used more generally. To find a particular integral for

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k_1\, f_1(x) + k_2\, f_2(x), \tag{50}$$

where k_1 and k_2 are constants, we can split the task up – finding the particular integral $g_1(x)$ that applies when just $f_1(x)$ is on the right-hand side, and the particular integral $g_2(x)$ that applies when just $f_2(x)$ is on the right-hand side. The principle of superposition then tells us that $k_1\, g_1(x) + k_2\, g_2(x)$ is a particular integral for equation (50).

Exercise 19

Find a particular integral for the differential equation

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 12\cos 2t + 10.$$

(Neither of the terms on the right-hand side satisfies the associated homogeneous equation, so this is not an exceptional case.)

Exercise 20

Find the general solutions of the following differential equations (starting from scratch with no complementary functions given).

(a) $u''(t) + 4u'(t) + 5u(t) = 5$ (b) $3\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - y = e^{2x}$

4 Initial conditions and boundary conditions

In Sections 2 and 3 you saw how to find the general solution of a homogeneous or inhomogeneous linear constant-coefficient second-order differential equation. In practice, however, we usually need to select a *particular solution* that satisfies certain additional conditions. This section explains how this is done.

In Unit 2 you saw that the general solution of a first-order differential equation contains *one* arbitrary constant, and that a single additional condition (called an initial condition) is enough to fix the value of this constant and hence determine the particular solution. In the case of a second-order differential equation, the general solution contains *two* arbitrary constants, and *two* additional conditions are required.

There are two types of additional conditions for second-order differential equations: *initial conditions* and *boundary conditions*. Problems involving such conditions are called *initial-value problems* and *boundary-value problems*, respectively, and are discussed in Subsections 4.1 and 4.2.

4.1 Initial-value problems

For a first-order differential equation, an initial condition is one that specifies the value of the dependent variable ($y = a$, say) at a given value of the independent variable ($t = t_0$); this is often written in the form $y(t_0) = a$.

For a second-order differential equation, the initial conditions specify the value of the dependent variable ($y = a$) *and* the value of its derivative ($dy/dy = b$), for the *same* given value of the independent variable ($t = t_0$), and they are often written in the form $y(t_0) = a$, $y'(t_0) = b$.

Very often, t_0 represents the initial time when a system is released and we are interested in the subsequent motion. But this is not essential; we could equally well be interested in the prior motion that leads up to given values of x and dx/dt at some final time t_0. Indeed, initial values may have nothing to do with time at all if the independent variable represents some other quantity such as position. The only essential point is that values of the dependent variable and its derivative must both be given at the *same* value of the independent variable.

Initial conditions arise naturally in many mechanical problems, where the initial values of the position x and velocity dx/dt are often specified. For example, we may know that a ball is thrown vertically upwards, at $t = 0$, from an initial position with an initial velocity.

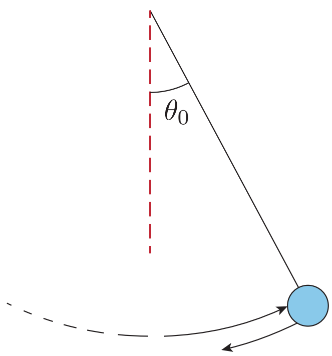

Figure 9 Possible initial conditions for a pendulum

The pendulum in Figure 9 gives another example. When the string of the pendulum makes its greatest angle θ_0 with the vertical, the pendulum changes its direction of swing and comes momentarily to rest. So if the pendulum changes direction at $t = t_0$, we have the initial conditions $\theta = \theta_0$ and $d\theta/dt = 0$ when $t = t_0$.

Definitions

- **Initial conditions** associated with a second-order differential equation with dependent variable y and independent variable x specify that y and dy/dx take values a and b, respectively, when x takes the value x_0. These conditions can be written as

$$y = a \text{ and } \frac{dy}{dx} = b \text{ when } x = x_0$$

  or as

$$y(x_0) = a, \quad y'(x_0) = b.$$

  The numbers x_0, a and b are often referred to as **initial values**.
- The combination of a second-order differential equation and initial conditions is called an **initial-value problem**.

The following example shows how initial conditions can be used to find the two arbitrary constants in the general solution, and hence determine a particular solution of a second-order differential equation.

Example 13

The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

is

$$y = Ce^x + De^{2x}, \tag{51}$$

where C and D are arbitrary constants (see Example 3). Find the particular solution that satisfies the initial conditions $y = 0$ and $dy/dx = 1$ when $x = 0$.

Solution

One of the initial conditions involves dy/dx, so we take the derivative of the general solution (51), getting

$$\frac{dy}{dx} = Ce^x + 2De^{2x}. \tag{52}$$

The initial conditions state that $y(0) = 0$, $y'(0) = 1$. Substituting $x = 0$, $y = 0$ into equation (51) gives

$$0 = Ce^0 + De^0 = C + D,$$

while substituting $x = 0$, $dy/dx = 1$ into equation (52) gives

$$1 = Ce^0 + 2De^0 = C + 2D.$$

Note that when you are checking a particular solution, you should check that it satisfies the initial or boundary conditions as well as the differential equation.

Solving these equations gives $C = -1$, $D = 1$, so the required particular solution is

$$y = -e^x + e^{2x}.$$

All the initial-value problems that you will meet in this module have *unique* solutions, so if you can find a solution that satisfies the initial conditions, then this is *the* solution.

Exercise 21

Solve the following initial-value problems.

(a) The general solution of $u''(t) + 9u(t) = 0$ is $u = C\cos 3t + D\sin 3t$, where C and D are arbitrary constants (see Exercise 4(b)). Find the particular solution that satisfies the initial conditions $u\left(\frac{\pi}{2}\right) = 0$, $u'\left(\frac{\pi}{2}\right) = 1$.

(b) The general solution of $u''(t) + 4u'(t) + 5u(t) = 5$ is $u = e^{-2t}(C\cos t + D\sin t) + 1$, where C and D are arbitrary constants (see Exercise 20(a)). Find the particular solution that satisfies the initial conditions $u(0) = 3$, $u'(0) = 1$.

4.2 Boundary-value problems

In initial-value problems, the two conditions used to select a particular solution both refer to the *same* value of the independent variable. But this need not be the case. If the independent variable is x, then we could have one condition for $x = x_1$ and another for $x = x_2$, say. Such conditions are called *boundary conditions*. Later in the module (Unit 12) you will meet equations called *partial differential equations* which explain a vast range of phenomena; in many cases, these equations lead to second-order differential equations supplemented by boundary conditions.

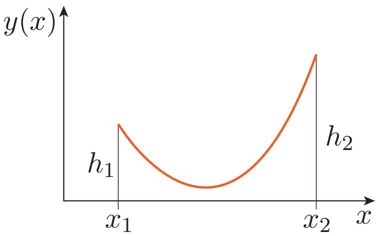

Figure 10 Finding the shape of a hanging chain with fixed ends

Boundary conditions arise, for example, in considering the shape of a chain of length L that is suspended between two points, with heights h_1 and h_2 (see Figure 10). If $y(x)$ is the height of the chain at a horizontal distance x from an origin, then the equation for $y(x)$ has to satisfy two boundary conditions: $y(x_1) = h_1$ and $y(x_2) = h_2$. This pair of boundary conditions gives the value of y at two different points. In other contexts, each boundary condition could specify the value of either y or dy/dx (or even a relationship between them).

The conditions are called 'boundary' conditions because (as in the chain example) they often refer to conditions at the endpoints x_1 and x_2 of an interval in which we want to explore the solutions of a differential equation. (This is not essential, however.)

Definitions

Consider a second-order differential equation with dependent variable y and independent variable x.

- **Boundary conditions** associated with such an equation specify the values of y (or dy/dx, or some combination of y and dy/dx) at two *different* values of x. For example, they could specify that $y(x_1) = y_1$ and $y(x_2) = y_2$. The numbers x_1, x_2, y_1 and y_2 are often referred to as **boundary values**.
- The combination of a second-order differential equation and boundary conditions is called a **boundary-value problem**.

The following example shows how boundary conditions can be used to find values for the two arbitrary constants that appear in the general solution of a second-order differential equation, and hence find a particular solution.

Example 14

The differential equation

$$\frac{d^2y}{dx^2} + 9y = 0$$

has general solution

$$y = C\cos 3x + D\sin 3x, \tag{53}$$

where C and D are arbitrary constants (see Exercise 4(b)). Find the particular solution that satisfies the boundary conditions $y = 0$ when $x = 0$ and $dy/dx = 1$ when $x = \frac{\pi}{3}$.

Solution

One of the boundary conditions involves dy/dx, so we need the derivative of the general solution (53):

$$\frac{dy}{dx} = -3C \sin 3x + 3D \cos 3x. \tag{54}$$

The boundary conditions state that $y(0) = 0$, $y'\left(\frac{\pi}{3}\right) = 1$. Substituting $x = 0$, $y = 0$ into equation (53) gives

$$0 = C \cos 0 + D \sin 0 = C,$$

so $C = 0$. Substituting $x = \frac{\pi}{3}$, $y' = 1$ and $C = 0$ into equation (54) gives

$$1 = 3D \cos \pi = -3D.$$

Therefore $C = 0$ and $D = -\frac{1}{3}$, so the required particular solution is

$$y = -\tfrac{1}{3} \sin(3x).$$

Unlike initial-value problems, some boundary-value problems may have *no solutions* even when the differential equation is linear and constant-coefficient, and has a continuous function on the right-hand side. The following example illustrates this point.

Example 15

The differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

has general solution $y = C \cos 2x + D \sin 2x$, where C and D are arbitrary constants (see Exercise 6(a)). Try to find a solution to the boundary-value problem based on this differential equation and the boundary conditions $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$.

Solution

Substituting each of the boundary conditions into the general solution in turn gives

$$0 = C \cos 0 + D \sin 0 = C,$$
$$1 = C \cos \pi + D \sin \pi = -C.$$

There is no solution for which $C = 0$ and $C = -1$, so there is no solution of the differential equation that satisfies the given boundary conditions.

An outcome of 'no solution' need not be unreasonable. If a physical system obeys a differential equation, and there is no solution consistent with a given set of boundary conditions, then those boundary conditions must be unrealistic. For example, the chain in Figure 10 cannot hang from points that are further apart than its fixed length.

It is also possible for boundary-value problems to have solutions that are *not unique*, as the following important example illustrates.

Example 16

At a given moment in time, the displacement y of an oscillating guitar string at a distance x from one of its ends satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

for a fixed constant k, which is proportional to the frequency of the oscillation (i.e. the pitch of sound produced). The string is held fixed at its ends, $x = 0$ and $x = L$, so that $y(x)$ satisfies the boundary conditions $y(0) = 0$ and $y(L) = 0$. Find the possible solutions to this boundary-value problem, and show that solutions can be found only if $k = \pi n/L$, where n is an integer.

Solution

The general solution of the differential equation is

$$y(x) = A\sin(kx) + B\cos(kx),$$

where A and B are arbitrary constants.

The boundary condition $y(0) = 0$ implies that $B = 0$, so $y(x) = A\sin(kx)$. In order to satisfy the boundary condition $y(L) = 0$, we must have $A\sin(kL) = 0$. This equation has the trivial solution $A = 0$, which corresponds to $y(x) = 0$. For $A \neq 0$, it gives the condition $\sin(kL) = 0$, which implies that $kL = n\pi$, where n is an integer. Hence $k = n\pi/L$ and we conclude that the displacement of the guitar string must be of the form

$$y(x) = A\sin\left(\frac{n\pi x}{L}\right). \tag{55}$$

When considering the possible sinusoidal shapes adopted by an oscillating guitar string, we can restrict n to the positive integers $n = 1, 2, 3, \ldots$. This is because $A\sin(-n\pi x/L) = -A\sin(n\pi x/L)$, so changing the sign of n is equivalent to changing the sign of A. The value $n = 0$ can also be omitted, because it is equivalent to taking $A = 0$.

Motions with different values of n are said to be different **modes of oscillation** of the string. The first three modes of oscillation are shown schematically in Figure 11; in each case, the largest excursions of the string from the equilibrium $y = 0$ position are indicated.

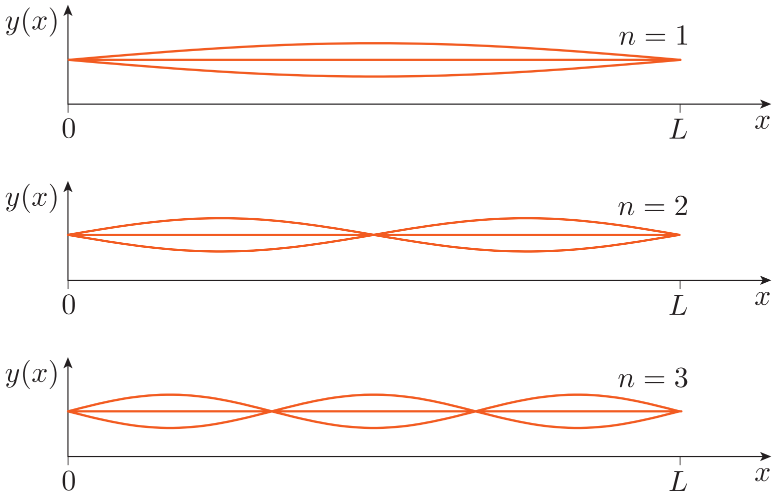

Figure 11 The first three modes of oscillation of a guitar string. These modes produce sounds of different pitches.

The fact that there are many different solutions satisfying the boundary conditions should not alarm you. This just means that the boundary conditions restrict the possible solutions, but extra information (in the form of initial conditions telling us how the guitar string is released) is needed to determine the precise motion that arises in a given situation.

Partial differential equations

The boundary conditions allow us to identify the possible modes, but further information is needed to predict the actual motion of the string. You will see how this works in Unit 12, when we discuss a more general type of differential equation, called a *partial differential equation*. The process of solving a partial differential equation often leads to a boundary-value problem for a second-order differential equation, and this is the main reason why boundary-value problems are important in physics and engineering.

An example arises in quantum mechanics, which can give rise to a differential equation very like that for a vibrating string. In this context, the fact that the constant k takes discrete values corresponds to the fact that the system has discrete energy levels, a phenomenon called the *quantisation of energy*.

Exercise 22

This question refers to the differential equation

$$u''(x) + 4u(x) = 0,$$

which has the general solution $u = C\cos 2x + D\sin 2x$, where C and D are arbitrary constants (see Exercise 6(a)).

Each part gives a set of additional conditions. State whether these are initial conditions or boundary conditions, and find the solution (or solutions) that satisfy both the differential equation and the additional conditions.

(a) $u(0) = 1$, $u'(0) = 0$ (b) $u(0) = 0$, $u\left(\frac{\pi}{2}\right) = 0$

(c) $u\left(\frac{\pi}{2}\right) = 0$, $u'\left(\frac{\pi}{2}\right) = 0$ (d) $u(-\pi) = 1$, $u\left(\frac{\pi}{4}\right) = 2$

5 Resonance

> The material in this section is non-assessable and will not be tested in continuous assessment or in the exam. However, we strongly advise students of physical science to study it, as these ideas will be used in higher-level modules.

In Subsection 2.4, we used the physical example of a damped harmonic oscillator to gain insights into the solutions of homogeneous linear constant-coefficient second-order differential equations. This final section does something similar for inhomogeneous equations.

Suppose that an object of mass m moves to and fro along the x-axis around an equilibrium position $x = 0$. The object has position $x(t)$ at time t. It experiences a force $-kx$ due to a spring, and a damping force $-\gamma\, dx/dt$ that opposes the motion of the object, where the constants m, k and γ are all positive. Up to this point, the system is just the damped harmonic oscillator considered earlier.

Now suppose that an additional time-dependent force $f(t)$ is applied to the object. This force is due to some external agency, and does not depend on the object's position or velocity. Then Newton's second law tells us that

$$-kx - \gamma\frac{dx}{dt} + f(t) = m\frac{d^2x}{dt^2},$$

which is equivalent to the inhomogeneous linear constant-coefficient second-order differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = f(t). \tag{56}$$

This system is known as a **forced damped harmonic oscillator**, and the external force is called the **driving force**.

A very important case occurs when the externally-applied driving force is a sinusoidal function of time. We therefore take

$$f(t) = F_0 \sin \omega t,$$

where F_0 and ω are positive constants. Under these circumstances, equation (56) can be written in the form

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega_0^2 x = a_0 \sin \omega t, \tag{57}$$

where $\Gamma = \gamma/2m$, $\omega_0 = \sqrt{k/m}$ and $a_0 = F_0/m$ are all positive constants. Here, ω_0 is the angular frequency of the simple harmonic oscillator (in the absence of damping or external forces). We include the subscript zero to distinguish ω_0 from the angular frequency ω of the driving force.

According to Procedure 2, the general solution of equation (57) is the sum of a complementary function $x_\text{c}(t)$ and a particular integral $x_\text{p}(t)$. The complementary function is the general solution of the associated homogeneous equation

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega_0^2 x = 0,$$

and you saw in Subsection 2.4 that this general solution contains an exponentially decaying factor $e^{-\Gamma t}$. Because of this factor, the complementary function represents a short-lived or transient contribution to the motion, which eventually dies away. Once this transient contribution has become negligible, the system settles down to a steady-state motion, given by the particular integral, which we focus on here.

This decay is a direct consequence of the damping force.

The required particular integral can be found using the methods of Subsection 3.2. We can substitute a trial solution

$$x(t) = p\cos(\omega t) + q\sin(\omega t)$$

into differential equation (57), and find p and q by matching the coefficients of $\cos(\omega t)$ and of $\sin(\omega t)$ on both sides of the equation. The method is identical to that used in Example 10.

The algebra gets a little messy, so we now introduce an alternative approach, which is more efficient in this case. The key ingredient is Euler's formula, which tells us that

$$a_0 e^{i\omega t} = a_0 \cos(\omega t) + i a_0 \sin(\omega t),$$

the imaginary part of which is equal to the right-hand side of equation (57). We therefore consider a differential equation that is closely related to equation (57), namely

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_0 e^{i\omega t}. \tag{58}$$

Because the right-hand side of this equation is complex, the function $z(t)$ in this equation must also be complex.

Substituting $z(t) = u(t) + i\,v(t)$ into equation (58), and equating real parts and imaginary parts on both sides, we get

$$\frac{d^2u}{dt^2} + 2\Gamma\frac{du}{dt} + \omega_0^2 u = a_0\cos(\omega t),$$
$$\frac{d^2v}{dt^2} + 2\Gamma\frac{dv}{dt} + \omega_0^2 v = a_0\sin(\omega t).$$

It follows that $v(t)$, which is the imaginary part of $z(t)$, satisfies the main equation of interest, equation (57). Our tactic will therefore be to solve equation (58) for $z(t)$ and then take its imaginary part; this is an effective shortcut because equation (58) is easier to solve than equation (57).

Substituting the trial solution $z = pe^{i\omega t}$ into equation (58), we get

$$\left[(i\omega)^2 + 2\Gamma(i\omega) + \omega_0^2\right] pe^{i\omega t} = a_0 e^{i\omega t},$$

from which we obtain the particular integral

$$z(t) = a_0\,\frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\Gamma\omega}.$$

Multiplying top and bottom by $\omega_0^2 - \omega^2 - 2i\Gamma\omega$ and using Euler's formula, this can be written as

$$z(t) = a_0\,\frac{\left[\cos(\omega t) + i\sin(\omega t)\right]\left[(\omega_0^2 - \omega^2) - 2i\Gamma\omega\right]}{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}. \tag{59}$$

The solution that we need is the imaginary part of this expression. Multiplying out the brackets and picking out the coefficient of i, we conclude that equation (57) has the particular integral

$$x(t) = a_0\,\frac{(\omega_0^2 - \omega^2)\sin(\omega t) - 2\Gamma\omega\cos(\omega t)}{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2},$$

which is of the expected form

$$x(t) = p\cos(\omega t) + q\sin(\omega t),$$

where

$$p = \frac{-2\Gamma\omega}{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2} \quad\text{and}\quad q = \frac{(\omega_0^2 - \omega^2)a_0}{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}.$$

The interpretation of this expression is clarified by writing it in the form

$$x(t) = A\sin(\omega t + \phi), \tag{60}$$

where A is the amplitude and ϕ is the phase constant. These can be found by using the argument that led to equations (31) and (32) in Subsection 2.4. We have $A = \sqrt{p^2 + q^2}$ and $\tan\phi = p/q$, so

$$A = \frac{a_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}} \tag{61}$$

and

$$\tan\phi = -\frac{2\Gamma\omega}{\omega_0^2 - \omega^2}. \tag{62}$$

Equation (60) gives the response of the system to the driving force once the transient motion associated with the complementary function has died away. Not surprisingly, the system performs a sinusoidal oscillation with the same angular frequency ω as the driving force. However, the amplitude A and phase constant ϕ behave in interesting ways. We focus on cases where the oscillator is underdamped ($\Gamma < \omega_0$).

The resonance behaviour illustrated in Figure 12 appears only if the oscillator is underdamped. A forced *overdamped* oscillator still performs sinusoidal oscillations, but its amplitude always decreases as ω increases.

Figure 12(a) shows the amplitude A as a function of the driving angular frequency ω for four different values of the damping parameter Γ. In each case, the amplitude is greatest when ω is equal to the natural angular frequency ω_0 of the corresponding undriven undamped oscillator. This effect appears as a peak in the graph, which becomes higher and narrower as the damping parameter Γ is reduced. This phenomenon is called **resonance**, and the angular frequency $\omega = \omega_0$ at which the response is greatest is called the **resonant angular frequency**.

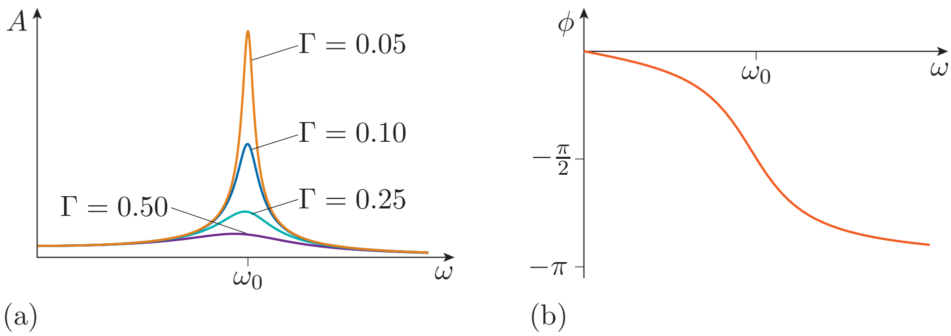

Figure 12 The response of a forced underdamped harmonic oscillator with $\omega_0 = 1$ as a function of the driving angular frequency ω:
(a) the amplitude A for $\Gamma = 0.50$, $\Gamma = 0.25$, $\Gamma = 0.10$ and $\Gamma = 0.05$;
(b) the corresponding phase constant ϕ

The phase constant ϕ is plotted in Figure 12(b). In general, the oscillation of the responding system lags behind the driving force, so ϕ is negative. The phase constant does not depend on the amplitude of the driving force, but it does depend on its angular frequency. At low angular frequencies, the lag is small. At the resonant angular frequency, the lag is a quarter of a cycle, and this grows to half a cycle at frequencies that are much larger than ω_0.

In plotting the graph in Figure 12(b), it was assumed that ϕ is a continuous function of ω. The arctan function was therefore allowed to extend beyond the range from $-\pi/2$ to $\pi/2$.

Exercise 23

Suppose that the displacement $x(t)$ of a forced damped harmonic oscillator obeys the inhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega_0^2 x = a_0 \cos(\omega t),$$

where Γ, ω_0 and a_0 are positive constants. Use the method based on equation (58) to find a particular integral for this equation, and hence obtain an expression for the amplitude as a function of ω.

Exercise 24

Use equation (61) to obtain simple formulas for the amplitude of oscillation of a forced underdamped oscillator in the following cases.

Express all your answers as simple fractions (involving a_0, ω_0, Γ and ω as necessary). You will need to make suitable approximations in the first two cases.

(a) $\omega \ll \omega_0$ (b) $\omega \gg \omega_0$ (c) $\omega = \omega_0$

Figure 13 A simple radio receiver, which uses the phenomenon of resonance to select which transmission to receive

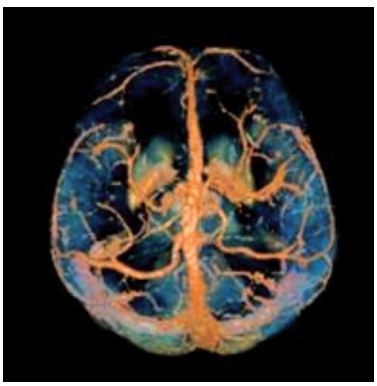

Figure 14 A false-colour MRI image of a brain of a healthy volunteer, emphasising the veins on top and the basal ganglia beneath

Examples of resonance effects

Resonance occurs when an oscillation is sustained by an external influence, and the amplitude of the oscillation peaks strongly at a particular angular frequency of the external influence.

A familiar example occurs when you push a swing. Swings are designed to have little friction, so Γ is small and there is a pronounced resonance effect. You can get a motion of large amplitude if you push at the same angular frequency as the natural angular frequency of the swing.

Simple radio receivers (Figure 13) use the phenomenon of resonance to select which radio station to receive. Every radio station works at a different 'carrier' frequency. A circuit inside the receiver obeys the same equation as the damped harmonic oscillator, with electrical current playing the role of displacement $x(t)$. Tuning the radio receiver changes the natural frequency of this circuit, bringing it into resonance with the frequencies of various radio stations. In a radio receiver, Γ is very small, so the receiver responds to only a very narrow band of frequencies, allowing a single station to be selected.

The image shown in Figure 14 is from a magnetic resonance imaging (MRI) machine. In this case, the oscillating quantity is provided by magnetic properties of atomic nuclei, and the resonant frequency depends on the strength of an applied magnetic field. Images are obtained by placing the patient in a strong *non-uniform* magnetic field, so that different parts of the body are characterised by different resonant frequencies.

Finally, when a light wave (or other electromagnetic wave) is shone on a material, it causes the electrons in the material to oscillate to and fro. In a classical model, the electrons behave like forced damped harmonic oscillators, with the driving force supplied by the light wave. Energy is transferred from the light wave to the oscillating electrons, so some of the light is absorbed. We can therefore use the equations of forced damped harmonic motion to model the absorption of light in materials.

The absorption of light by single atoms is best described by quantum mechanics. Rather remarkably, however, resonance peaks in absorption are still predicted. Figure 15 is an image of the spectrum of sunlight. Among the dark lines there are lines due to absorption of light by helium in the solar atmosphere. These lines are found where the frequency of vibration of the sunlight is in resonance with frequencies of helium atoms. This approach demonstrated the existence of helium many years before it was isolated on the Earth.

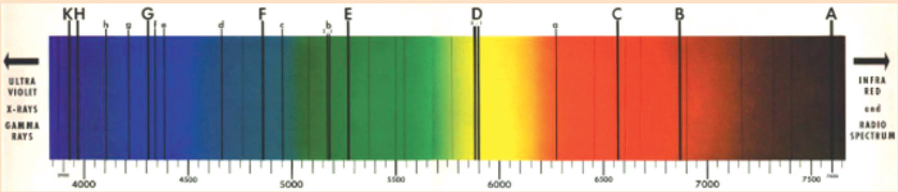

Figure 15 Dark lines in the spectrum of sunlight, due to a resonance with atoms in the atmosphere of the Sun, which demonstrates the existence of helium

Learning outcomes

After studying this unit, you should be able to do the following.

- Understand and use the terminology relating to linear constant-coefficient second-order differential equations.
- Understand the key role of the principle of superposition in the solution of linear constant-coefficient second-order differential equations.
- Obtain the general solution of a homogeneous linear constant-coefficient second-order differential equation using the solutions of its auxiliary equation.
- Use the method of undetermined coefficients to find a particular integral for an inhomogeneous linear constant-coefficient second-order differential equation with certain simple forms of right-hand-side function.
- Obtain the general solution of an inhomogeneous linear constant-coefficient second-order differential equation by combining its complementary function with a particular integral.
- Use the general solution together with a pair of initial or boundary conditions to obtain, when possible, a particular solution of a linear constant-coefficient second-order differential equation.

Solutions to exercises

Solution to Exercise 1

(a) In equations (i)–(vi), the (dependent, independent) variable pairs are all (y, x). In equations (vii), (viii) and (ix) they are (t, θ), (x, t) and (x, t), respectively.

(b) Equations (i), (ii), (iii), (iv), (vii), (viii) and (ix) are linear. (Equations (v) and (vi) are non-linear.)

(c) All of the linear equations are constant-coefficient except for (iv). So the linear constant-coefficient equations are (i), (ii), (iii), (vii), (viii) and (ix).

(d) Of the linear constant-coefficient equations, only (iii) and (ix) are homogeneous.

Solution to Exercise 2

(a) $\lambda^2 - 5\lambda + 6 = 0$

(b) $\lambda^2 - 9 = 0$

(c) $\lambda^2 + 2\lambda = 0$

Solution to Exercise 3

(a) The auxiliary equation is $\lambda^2 + 5\lambda + 6 = 0$. This can be factorised as $(\lambda + 2)(\lambda + 3) = 0$, giving the roots $\lambda_1 = -2$ and $\lambda_2 = -3$. The general solution is therefore

$$y = Ce^{-2x} + De^{-3x},$$

where C and D are arbitrary constants.

(b) The auxiliary equation is $2\lambda^2 + 3\lambda = 0$. This can be factorised as $\lambda(2\lambda + 3) = 0$, so its roots are $\lambda_1 = 0$ and $\lambda_2 = -\frac{3}{2}$. The general solution is therefore

Recall that $e^0 = \exp(0) = 1$.

$$y = Ce^0 + De^{-3x/2} = C + De^{-3x/2},$$

where C and D are arbitrary constants.

(c) The auxiliary equation is $\lambda^2 - 4 = 0$, i.e. $\lambda^2 = 4$, so its roots are $\lambda_1 = -2$ and $\lambda_2 = 2$. The general solution is therefore

$$z = Ce^{-2u} + De^{2u},$$

where C and D are arbitrary constants. (This differential equation is a special case of equation (14) discussed in Subsection 2.1.)

Solution to Exercise 4

(a) The auxiliary equation is $\lambda^2 + 4\lambda + 8 = 0$, which has solutions

$$\lambda = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i.$$

The general solution is therefore

$$y = e^{-2x}(C \cos 2x + D \sin 2x).$$

(b) The auxiliary equation is $\lambda^2 + 9 = 0$, which has solutions

$$\lambda = \pm 3i.$$

These are of the form $\alpha \pm \beta i$, where $\alpha = 0$ and $\beta = 3$.

The general solution is therefore

$$\theta = e^0(C \cos 3t + D \sin 3t) = C \cos 3t + D \sin 3t.$$

Of course, this is one of the simple cases discussed at the beginning of this section: it corresponds to simple harmonic motion.

Solution to Exercise 5

(a) The auxiliary equation is $\lambda^2 + 2\lambda + 1 = 0$, which can be factorised as $(\lambda + 1)^2 = 0$, giving equal roots $\lambda_1 = \lambda_2 = -1$. The general solution is therefore

$$y = (C + Dx)e^{-x},$$

where C and D are arbitrary constants.

(b) The auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$, which can be factorised as $(\lambda - 2)^2 = 0$, giving equal roots $\lambda_1 = \lambda_2 = 2$. The general solution is therefore

$$s = (C + Dt)e^{2t},$$

where C and D are arbitrary constants.

Solution to Exercise 6

In all cases, C and D are arbitrary constants.

(a) The auxiliary equation is $\lambda^2 + 4 = 0$, which has solutions $\lambda = \pm 2i$. The general solution is therefore

$$y = C \cos 2x + D \sin 2x.$$

(You could also have written down this general solution directly from equation (18).)

(b) The auxiliary equation is $\lambda^2 - 9 = 0$, which has solutions $\lambda = \pm 3$. The general solution is therefore

$$y = Ce^{3x} + De^{-3x}.$$

(You could also have written down this general solution directly from equation (15).)

(c) The auxiliary equation is $\lambda^2 + 2\lambda = 0$, which has solutions $\lambda_1 = 0$ and $\lambda_2 = -2$. The general solution is therefore

$$y = C + De^{-2x}.$$

(d) The auxiliary equation is $\lambda^2 - 2\lambda + 1 = 0$, which has solutions $\lambda_1 = \lambda_2 = 1$. The general solution is therefore

$$y = (C + Dx)e^x.$$

(e) The auxiliary equation is $\lambda^2 + 4\lambda + 29 = 0$, which has solutions

$$\lambda = \frac{-4 \pm \sqrt{16 - 116}}{2} = -2 \pm 5i.$$

The general solution is therefore

$$y = e^{-2x}(C\cos 5x + D\sin 5x).$$

(f) The auxiliary equation is $\lambda^2 - 6\lambda + 8 = 0$, which factorises as $(\lambda - 4)(\lambda - 2) = 0$ and has solutions $\lambda_1 = 4$ and $\lambda_2 = 2$. The general solution is therefore

$$u = Ce^{4x} + De^{2x}.$$

Solution to Exercise 7

(a) The auxiliary equation is

$$3\lambda - 1 - 2\lambda^2 = 0,$$

or equivalently,

$$2\lambda^2 - 3\lambda + 1 = 0,$$

which factorises as

$$(2\lambda - 1)(\lambda - 1) = 0.$$

(b) The two solutions of the auxiliary equation are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, so the general solution of the differential equation is

$$y = Ce^{x/2} + De^x,$$

where C and D are arbitrary constants.

Solution to Exercise 8

(a) The auxiliary equation is

$$\lambda^2 + 2\lambda + 2 = 0.$$

This has solutions $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$, so the general solution is

$$y = e^{-x}(C\cos x + D\sin x).$$

(b) The auxiliary equation is

$$\lambda^2 - 16 = 0.$$

This has solutions $\lambda_1 = 4$ and $\lambda_2 = -4$, so the general solution is

$$y = Ce^{4x} + De^{-4x}.$$

(c) The auxiliary equation is

$$\lambda^2 - 4\lambda + 4 = 0.$$

This has solutions $\lambda_1 = \lambda_2 = 2$, so the general solution is

$$y = (C + Dx)e^{2x}.$$

(d) The auxiliary equation is

$$\lambda^2 + 3\lambda = 0.$$

This has solutions $\lambda_1 = 0$ and $\lambda_2 = -3$, so the general solution is

$$\theta = C + De^{-3t}.$$

Solution to Exercise 9

The auxiliary equation $\lambda^2 + 4k\lambda + 4 = 0$ can be solved using the formula method to give $\lambda = -2k \pm 2\sqrt{k^2 - 1}$. So there are complex conjugate solutions, leading to a general solution involving sines and cosines, when $k^2 < 1$, i.e. when $-1 < k < 1$.

Solution to Exercise 10

(a) The differential equation can be written in the form

$$\frac{d^2\theta}{dt^2} + 2\Gamma\frac{d\theta}{dt} + \omega^2\theta = 0,$$

where

$$\Gamma = \frac{\gamma}{2m} = \frac{0.016}{2 \times 0.80} = 0.01,$$

and the angular frequency of the corresponding undamped oscillator is

$$\omega = \sqrt{\frac{mg}{ml}} = \sqrt{\frac{9.8}{2.0}} = 2.21.$$

We therefore have $\Gamma < \omega$, and the oscillation is underdamped.

(b) The angular frequency of the damped pendulum is

$$\Omega = \sqrt{\omega^2 - \Gamma^2} = \sqrt{2.21^2 - 0.01^2} = 2.21.$$

This is typical: the angular frequency is not very sensitive to damping when $\Gamma \ll \omega$.

We therefore have

$$T = \frac{2\pi}{\Omega} = 2.84,$$

so the period of oscillation is 2.8 seconds.

(c) The amplitude is given by $Ae^{-\Gamma t}$, where A is a constant. If the amplitude is 0.2 radians at time zero, the amplitude at time $t = 100$ is

$$Ae^{-\Gamma t} = 0.20e^{-0.01\times 100} = 0.2e^{-1} = 0.074 \quad \text{(in radians)}.$$

Solution to Exercise 11

The condition for critical damping is $\Gamma = \omega = \sqrt{k/m}$, so in this case $\Gamma = \sqrt{4} = 2$. The general solution is therefore

$$x(t) = (C + Dt)e^{-2t}.$$

Solution to Exercise 12

(a) The associated homogeneous equation is

$$\frac{d^2y}{dx^2} + 4y = 0.$$

The complementary function (see Exercise 6(a)) is

$$y_c = C\cos 2x + D\sin 2x.$$

Trying a solution of the form $y_p = p$, where p is a constant, in the original equation $d^2y/dx^2 + 4y = 8$ gives $0 + 4p = 8$, so $p = 2$. Thus a particular integral is

$$y_p = 2.$$

By Procedure 2, the general solution is

$$y = C\cos 2x + D\sin 2x + 2.$$

(b) The associated homogeneous equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

The complementary function (see Example 3) is

$$y_c = Ce^x + De^{2x}.$$

Trying a solution of the form $y_p = p$ in the original equation $d^2y/dx^2 - 3dy/dx + 2y = 6$ gives $0 - 0 + 2p = 6$, so $p = 3$. Thus a particular integral is

$$y_p = 3.$$

By Procedure 2, the general solution is

$$y = Ce^x + De^{2x} + 3.$$

Solution to Exercise 13

(a) Substituting $y = p_1x + p_0$ and its derivatives into the differential equation gives

$$0 - 2p_1 + 2(p_1x + p_0) = 2p_1x + (2p_0 - 2p_1) = 2x + 3.$$

Equating the coefficients of x gives $p_1 = 1$, and equating the constant terms gives $p_0 = \frac{5}{2}$. Therefore a particular integral is

$$y_p = x + \tfrac{5}{2}.$$

(b) Substituting $y = p_1x + p_0$ and its derivatives into the differential equation gives

$$0 + 2p_1 + (p_1x + p_0) = p_1x + (2p_1 + p_0) = 2x.$$

Equating the coefficients of x gives $p_1 = 2$, and equating the constant terms gives $p_0 = -4$, so a particular integral is

$$y_p = 2x - 4.$$

Solution to Exercise 14

We try $y = p_2t^2 + p_1t + p_0$, which has derivatives $y' = 2p_2t + p_1$ and $y'' = 2p_2$. Substituting these into the differential equation gives

$$\begin{aligned} 2p_2 - (p_2t^2 + p_1t + p_0) &= -p_2t^2 - p_1t + (2p_2 - p_0) \\ &= t^2. \end{aligned}$$

Hence, separately equating coefficients of t^2, t and 1, we get $p_2 = -1$, $p_1 = 0$, $p_0 = -2$, so a particular integral is

$$y_\text{p} = -t^2 - 2.$$

Solution to Exercise 15

We try a solution of the form $y = pe^{-x}$, which has derivatives $dy/dx = -pe^{-x}$ and $d^2y/dx^2 = pe^{-x}$. Substituting these into the differential equation gives

$$2pe^{-x} + 2pe^{-x} + pe^{-x} = 5pe^{-x} = 2e^{-x}.$$

Hence $p = \frac{2}{5}$, and a particular integral is

$$y_\text{p} = \tfrac{2}{5}e^{-x}.$$

Solution to Exercise 16

We try $y = p\cos 3t + q\sin 3t$, which has derivatives

$$\frac{dy}{dt} = -3p\sin 3t + 3q\cos 3t, \quad \frac{d^2y}{dt^2} = -9p\cos 3t - 9q\sin 3t.$$

Substituting into the differential equation gives

$$\begin{aligned} &(-9p\cos 3t - 9q\sin 3t) - (-3p\sin 3t + 3q\cos 3t) \\ &\quad = -(9p + 3q)\cos 3t + (3p - 9q)\sin 3t \\ &\quad = \cos 3t + \sin 3t. \end{aligned}$$

Hence we have a pair of simultaneous equations

$$\begin{aligned} -9p - 3q &= 1, \\ 3p - 9q &= 1. \end{aligned}$$

Adding three times the second equation to the first shows that $q = -\frac{4}{30} = -\frac{2}{15}$, hence $p = -\frac{1}{15}$. A particular integral is thus

$$y_\text{p} = -\tfrac{1}{15}\cos 3t - \tfrac{2}{15}\sin 3t.$$

Solution to Exercise 17

(a) Try $y = pe^{3x}$.

(b) Try $y = p\cos 3x + q\sin 3x$.

Solution to Exercise 18

(a) The complementary function is

$$\theta_c = C\cos 2t + D\sin 2t.$$

To find a particular integral, try $\theta = p_1 t + p_0$. Substituting this and its derivatives into the differential equation gives

$$4(p_1 t + p_0) = 2t.$$

Hence $p_1 = \frac{1}{2}$, $p_0 = 0$, and a particular integral is

$$\theta_p = \tfrac{1}{2}t.$$

Therefore the general solution is

$$\theta = C\cos 2t + D\sin 2t + \tfrac{1}{2}t,$$

where C and D are arbitrary constants.

(b) The complementary function is

$$y_c = C\cos 2x + D\sin 2x.$$

The right-hand-side function is $10\sin 3x$, so we try $y = p\cos 3x + q\sin 3x$. The derivatives are

$$\frac{dy}{dx} = -3p\sin 3x + 3q\cos 3x,$$
$$\frac{d^2y}{dx^2} = -9p\cos 3x - 9q\sin 3x.$$

Substituting into the differential equation gives

$$(-9p\cos 3x - 9q\sin 3x) + 4(p\cos 3x + q\sin 3x)$$
$$= -5p\cos 3x - 5q\sin 3x = 10\sin 3x,$$

so $p = 0$ and $q = -2$, and a particular integral is

$$y_p = -2\sin 3x.$$

Therefore the general solution is

$$y = C\cos 2x + D\sin 2x - 2\sin 3x,$$

where C and D are arbitrary constants.

(c) The complementary function is

$$y_c = Ce^{-2x} + De^{2x}.$$

The right-hand-side function is $15e^{-x}$, so we try $y = pe^{-x}$. The derivatives are

$$\frac{dy}{dx} = -pe^{-x}, \quad \frac{d^2y}{dx^2} = pe^{-x}.$$

Substituting for d^2y/dx^2 and y into the differential equation gives

$$pe^{-x} - 4pe^{-x} = -3pe^{-x} = 15e^{-x},$$

so $p = -5$, and a particular integral is

$$y_p = -5e^{-x}.$$

Therefore the general solution is

$$y = Ce^{-2x} + De^{2x} - 5e^{-x},$$

where C and D are arbitrary constants.

Solution to Exercise 19

We split the task of finding a particular integral into two parts, by first finding particular integrals for

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 12\cos 2t \tag{63}$$

and

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 10. \tag{64}$$

In equation (63), the term $12\cos 2t$ on the right-hand side suggests the trial solution $x = p\cos 2t + q\sin 2t$. This has derivatives

$$\frac{dx}{dt} = -2p\sin 2t + 2q\cos 2t, \quad \frac{d^2x}{dt^2} = -4p\cos 2t - 4q\sin 2t.$$

Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}&2(-4p\cos 2t - 4q\sin 2t) + 3(-2p\sin 2t + 2q\cos 2t) + 2(p\cos 2t + q\sin 2t)\\ &\quad = 6(q-p)\cos 2t - 6(p+q)\sin 2t.\end{aligned}$$

Equating this to $12\cos 2t$ from the right-hand side of equation (63) gives $p + q = 0$, $q - p = 2$. Hence $p = -1$, $q = 1$, and a particular integral is

$$x_{\mathrm{p}} = -\cos 2t + \sin 2t.$$

Now consider equation (64). In this case we try a constant function $x = p_0$. Substituting into the differential equation gives $2p_0 = 10$, so $p_0 = 5$, and a particular integral is

$$x_{\mathrm{p}} = 5.$$

Therefore, using the principle of superposition, a particular integral for the original differential equation is

$$x_{\mathrm{p}} = -\cos 2t + \sin 2t + 5.$$

Solution to Exercise 20

(a) The associated homogeneous equation has auxiliary equation

$$\lambda^2 + 4\lambda + 5 = 0,$$

which has solutions

$$\lambda = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

So the complementary function is

$$u_{\mathrm{c}} = e^{-2t}(C\cos t + D\sin t).$$

To find a particular integral, try $u = p_0$. Substituting gives $5p_0 = 5$. Hence $p_0 = 1$, and a particular integral is

$$u_{\text{p}} = 1.$$

Therefore the general solution is

$$u = e^{-2t}(C\cos t + D\sin t) + 1,$$

where C and D are arbitrary constants.

(b) The associated homogeneous equation has auxiliary equation

$$3\lambda^2 - 2\lambda - 1 = (3\lambda + 1)(\lambda - 1) = 0,$$

which has solutions $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{3}$. So the complementary function is

$$y_{\text{c}} = Ce^{x} + De^{-x/3}.$$

The right-hand side function is e^{2x}, so to find a particular integral, we try $y = pe^{2x}$. The derivatives are

$$\frac{dy}{dx} = 2pe^{2x}, \quad \frac{d^2y}{dx^2} = 4pe^{2x}.$$

Substituting gives

$$3(4pe^{2x}) - 2(2pe^{2x}) - pe^{2x} = 7pe^{2x} = e^{2x}.$$

Hence $p = \frac{1}{7}$, and a particular integral is

$$y_{\text{p}} = \tfrac{1}{7}e^{2x}.$$

Therefore the general solution is

$$y = Ce^{x} + De^{-x/3} + \tfrac{1}{7}e^{2x},$$

where C and D are arbitrary constants.

Solution to Exercise 21

(a) The derivative of the general solution $u = C\cos 3t + D\sin 3t$ is

$$u' = -3C\sin 3t + 3D\cos 3t.$$

Remember that $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$.

Substituting the initial condition $t = \frac{\pi}{2}$, $u = 0$ into the general solution gives $D = 0$. Substituting the initial condition $t = \frac{\pi}{2}$, $u' = 1$ into the derivative gives $C = \frac{1}{3}$. Hence the required particular solution is

$$u = \tfrac{1}{3}\cos 3t.$$

(b) The derivative of the general solution $u = e^{-2t}(C\cos t + D\sin t) + 1$ is

$$\begin{aligned} u' &= -2e^{-2t}(C\cos t + D\sin t) + e^{-2t}(-C\sin t + D\cos t) \\ &= e^{-2t}[(D - 2C)\cos t - (2D + C)\sin t]. \end{aligned}$$

Substituting the initial condition $u = 3$, $t = 0$ into the general solution gives $C = 2$. Substituting the initial condition $u' = 1$, $t = 0$ into the derivative gives $D - 2C = 1$, so $D = 5$. So the required particular solution is

$$u = e^{-2t}(2\cos t + 5\sin t) + 1.$$

Solution to Exercise 22

(a) The derivative of the general solution $u = C\cos 2x + D\sin 2x$ is

$$u' = -2C\sin 2x + 2D\cos 2x.$$

This part specifies an initial-value problem.

The initial condition $u(0) = 1$ gives $C = 1$. The initial condition $u'(0) = 0$ gives $D = 0$. The required solution is therefore

$$u = \cos 2x.$$

(b) This is a boundary-value problem.

The boundary condition $u(0) = 0$ gives $C = 0$. The boundary condition $u\left(\frac{\pi}{2}\right) = 0$ gives $C = 0$ also. D therefore remains arbitrary, so there is an infinite number of solutions, of the form

$$u = D\sin 2x.$$

(c) This is an initial-value problem.

The initial condition $u\left(\frac{\pi}{2}\right) = 0$ gives $C = 0$. The initial condition $u'\left(\frac{\pi}{2}\right) = 0$ gives $D = 0$. The required solution is therefore the zero function

$$u = 0.$$

(d) This is a boundary-value problem.

The boundary condition $u(-\pi) = 1$ gives $C = 1$. The boundary condition $u\left(\frac{\pi}{4}\right) = 2$ gives $D = 2$. The required solution is therefore

$$u = \cos 2x + 2\sin 2x.$$

Solution to Exercise 23

The differential equation given in this question is the real part of equation (58), so its solution is the real part of equation (59). Hence a suitable particular integral in this case is

$$x(t) = a_0\,\frac{(\omega_0^2 - \omega^2)\cos(\omega t) + 2\Gamma\omega\sin(\omega t)}{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}.$$

Using equation (31), the amplitude is the square root of the sum of the squares of the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, and so is given by

$$A = \left[a_0^2\,\frac{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}{\left((\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2\right)^2}\right]^{1/2} = \frac{a_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}}.$$

This is the same as equation (61), which is not surprising. Changing the right-hand side of the equation from $a_0\sin(\omega t)$ to $a_0\cos(\omega t)$ corresponds to changing the phase constant of the external force (which depends on the origin chosen for time). It is to be expected that this does not affect the amplitude of the steady-state forced damped oscillations.

Solution to Exercise 24

(a) Considering the expression

$$A = \frac{a_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\Gamma^2\omega^2}}$$

when $\omega \ll \omega_0$, we can neglect ω^2 in comparison with ω_0^2. We can also neglect $4\Gamma^2\omega^2$ in comparison with ω_0^4 (because $\Gamma < \omega_0$ for an underdamped oscillator). We therefore have the approximation

$$A \simeq \frac{a_0}{\omega_0^2} \quad \text{for } \omega \ll \omega_0.$$

In this case, the amplitude is a constant that is independent of the angular frequency of the external force or the damping parameter of the oscillator.

(b) When $\omega \gg \omega_0$, we can neglect ω_0^2 in comparison with ω^2. In the underdamped case, $\Gamma < \omega_0$ and we can also neglect $4\Gamma^2\omega^2$ in comparison with ω^4. We therefore have the approximation

$$A \simeq \frac{a_0}{\omega^2} \quad \text{for } \omega \gg \omega_0.$$

In this case, the amplitude decreases steadily as the angular frequency of the external force increases. The sluggish system is unable to keep up with the fast oscillations of the external force.

(c) Substituting $\omega = \omega_0$ into the general formula for the amplitude gives

$$A = \frac{a_0}{2\Gamma\omega_0}.$$

This is the amplitude at the peak of resonance. It increases without limit as the damping parameter Γ is decreased.

Acknowledgements

Grateful acknowledgement is made to the following sources:

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